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DECOMPOSITION OF TIME-SCALES IN LINEAR SYSTEMS USING DOMINANT EIGENSPACE POWER ITERATIONS AND MATCHED ASYMPTOTIC EXPANSIONS

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Many control theory concepts are valid for any system order, however, their actual use is limited to low order models. Large scale systems result not only in a formidable amount of computation, but also in ill-conditioned initial and two point boundary value problems. The interaction of fast and slow phenomena in high-order systems results in stiff numerical problems which require expensive integration routines. The singular perturbations approach to decomposing fast and slow phenomena involves using a time-scale separation technique. In this case a reduced order "steady state" and "boundary layer"

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solutions are obtained from a high order model. Control designs and simulations for the high order model are then carried out on the reduced order subsystems.

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DECOMPOSITION OF TIME-SCALES IN LINEAR SYSTEMS USING DOMINANT EIGENSPACE POWER ITERATIONS AND MATCHED ASYMPTOTIC EXPANSIONS

by

R. G. Phillips and P. V. Kokotovic

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1. EIGENSTRUCTURE DECOMPOSITION OF TIME SCALES IN LINEAR-TIME INVARIANT SYSTEMS

A. Introduction

Many control theory concepts are valid for any system order, however, their actual use is limited to low order models. Large scale systems result not only in a formidable amount of computation, but also in ill-conditioned initial and two point boundary value problems. The interaction of fast and slow phenomena in high-order systems results in stiff numerical problems which require expensive integration routines. The singular perturbations approach to decomposing fast and slow phenomena involves using a time-scale separation technique. In this case a reduced order "steady state" and "boundary layer" solutions are obtained from a high order model. Control designs and simulations for the high order model are then carried out on the reduced order subsystems.

It is the purpose of this chapter to unify and extend the results of previous authors [1-6] and attempt to provide a sense of completeness to the theory of time-scale separation in linear systems. Our concern here is the linear time invariant homogeneous system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad \begin{matrix} y(t_o) = y_o \\ z(t_o) = z_o \end{matrix} \tag{1.1}$$

$$y \in R^N , \quad z \in R^N$$

and transform it into either the form

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_\infty & B \\ 0 & B_\infty \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} \tag{1.2}$$

and/or the form

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_\infty^* & 0 \\ C & D_\infty^* \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} \tag{1.3}$$

where

$$\inf|\sigma(D_\infty)| > \sup|\sigma(A_\infty)| \tag{1.4}$$

and

$$\inf|\sigma(D_\infty^*)| > \sup|\sigma(A^*)| \tag{1.5}$$

and any such system (1.1) which has this property is said to satisfy the two-time-scale property for dimensions N and M.

In Section B, earlier methods of time-scale decomposition are presented. A power iteration method for computing the dominant left eigenspace of a matrix is used to show the equivalence and convergence criteria of the past schemes. This method is used for transforming (1.1) into (1.2) satisfying (1.4).

Section C uses a power iteration for computing the dominant right eigenspace of a matrix to show the existence and convergence of the transformation necessary to transform (1.1) into (1.3) satisfying (1.5).

Section D shows the duality of the transformations by exploring the eigenstructure of a matrix and its transpose.

Section E completes the block diagonalizations of (1.2) and (1.3) and identifies "fast" and "slow" components of our original state vectors. The explicit invertibility of our transformation matrices is shown. This becomes very important in later chapters.

In Section F we consider the problem of properly ordering the states. Finally, in Section G, we give an illustrative example.

B. Earlier Methods

In this section we present two earlier methods for transforming (1.1) into (1.2), and give there iterative schemes.

i) Quasi steady state method [6]

This method was motivated by singularly perturbed models. Here and throughout the remainder, it is assumed that the states are ordered and N and M are picked such that D^{-1} exists. This assumption is standard in studies of singularly perturbed systems, and, as will become apparent later, this assumption is not restrictive.

If the eigenvalues of D are such that the real parts are large and negative, then the homogeneous solution of z converges to a steady-state rapidly. If this convergence is assumed to be instantaneous, then $\dot{z} = 0$ and this quasi steady state assumption yields

$$z_s = -D^{-1}Cy_s. \tag{1.6}$$

Next we try to remove this slow part of z by introducing

$$\eta_1 = z + D^{-1}Cx \tag{1.7}$$

which transforms (1.1) into

$$\begin{bmatrix} \dot{y} \\ \dot{\eta}_1 \end{bmatrix} = \begin{bmatrix} (A-BD^{-1}C) & B \\ D^{-1}C(A-BD^{-1}C) & D+D^{-1}CB \end{bmatrix} \begin{bmatrix} y \\ \eta_1 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & B \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} y \\ \eta_1 \end{bmatrix}. \tag{1.8}$$

Repeating steps (1.6) and (1.7) k times results in the following

$$\eta_k = \eta_{k-1} + D_{k-1}^{-1}C_{k-1}y \qquad \eta_o = z \tag{1.9}$$

$$\begin{bmatrix} \dot{y} \\ \dot{\eta}_k \end{bmatrix} = \begin{bmatrix} A_k & B \\ C_k & D_k \end{bmatrix} \begin{bmatrix} y \\ \eta_k \end{bmatrix} \tag{1.10}$$

where the subsystem matrices are defined as

$$A_k = A_{k-1} - BD_{k-1}^{-1}C_{k-1} \qquad A_o = A \tag{1.11}$$

$$C_k = D_{k-1}^{-1}C_{k-1}A_k \qquad C_o = C \tag{1.12}$$

$$D_k = D_{k-1} + D_{k-1}^{-1}C_{k-1}B \quad D_o = D. \tag{1.13}$$

Experimental results, motivated by singular perturbations, have converged to the form (1.2) satisfying spectral property (1.4).

ii) Algebraic Riccati equation method [3]

In [2,3,4,7], the transformation of the form

$$\eta = z + Py \tag{1.14}$$

is proposed in an attempt to transform (1.1) into (1.2). By applying (1.14) to (1.1), it becomes

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A-BP & B \\ C-DP+PA-PBP & D+PB \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}. \tag{1.15}$$

The problem is to find the solution P to the Riccati type equation

$$R(P) = C - DP + PA - PBP = 0 \tag{1.16}$$

such that A-BP and D+PB have the spectral properties (1.4). Such spectrum dependent solutions have been referred to as "dichotomic" [8]. We will throughout the rest of this report continue to refer to this solution by this label.

Earlier work by [4] and more recent work by [3] have resulted in the following iterative recursion formula for obtaining the dichotomic

solution to (1.16)

$$P_{k+1} = P_k + (D + P_k B)^{-1} \cdot R(P_k) \tag{1.17}$$
$$P_o = D^{-1} C$$

which gives the subsystem matrices at each k as

$$A_k = A - BP_{k-1} \qquad A_o = A \tag{1.18}$$

$$D_k = D + P_{k-1}B \qquad D_o = D \tag{1.19}$$

$$C_k = R(P_{k-1}) = C - DP_{k-1} + P_{k-1}A - P_{k-1}BP_{k-1} \qquad C_o = C. \tag{1.20}$$

We now give a lemma which establishes a convergence criterion for (1.17). In the process, we shall show that (1.11)-(1.13) and (1.18)-(1.20) are equivalent at every iterate.

<u>Lemma 1</u>: Given (1.1), if the spectrum is concentrated in two groups of M and N such that

$$\sup_{i=1,N}|\lambda_i| \quad \inf_{j=1,M}|\lambda_j| . \tag{1.21}$$

Then under mild restrictions† on the initial iterate P_o, (1.17) will converge to the dichotomic solution of (1.16) at a convergence rate of ε^k, where

$$\varepsilon = \frac{\sup|\lambda_i| \quad i \in 1,N}{\inf|\lambda_j| \quad j \in 1,M} . \tag{1.22}$$

<u>Proof</u>: The well known power iteration method [9,10] for computing a M dimensional basis for the dominant left eigenspace of (1.1) is of the form

$$[M_k \quad N_k] = R_k [M_{k-1} \quad N_{k-1}] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.23}$$

†As in Ref.

R_k is a nonsingular m×m scaling matrix used, for example, to keep the rows of $[M_k\ N_k]$ strongly independent and the individual components within a practical range of computation [9]. Many methods have been proposed for selecting the sequence of R_k's and the interested reader is referred to [9] and [11]. The analytical convergence of (1.23), however, is independent of R_k. Thus, under condition (1.21) and mild conditions on $[M_o\ N_o]$, it is known that (1.23) converges to the dominant m-dimensional left eigenspace of (1.1).

Expressing the common iterates as

$$M_k = M_{k-1}A + N_{k-1}C \qquad M_o = C \tag{1.24}$$

$$N_k^{-1} = (D + N_{k-1}^{-1}M_{k-1}B)^{-1}N_{k-1}^{-1} \qquad N_o^{-1} = D^{-1}. \tag{1.25}$$

Expressing (1.24) and (1.25) as a product,

$$N_k^{-1}M_k = (D + N_{k-1}^{-1}M_{k-1}B)^{-1}(C + N_{k-1}^{-1}M_{k-1}A) \tag{1.26}$$

letting $L_k = N_k^{-1}M_k$ gives (1.26) as

$$L_k = (D + L_{k-1}B)^{-1}(C + L_{k-1}A) \qquad L_o = D^{-1}C \tag{1.27}$$

which is equivalent to (1.17) $\forall k \geq 0$.

To show the solution is dichotomic, P is of the form

$$P = N^{-1}M. \tag{1.28}$$

Without loss of generality we can take $[M\ \ N] = [V_1\ V_2]$ whre $[V_1\ V_2]$ are the M left generalized eigenvectors corresponding to the M dominant eigenvalues, thus

$$P = V_2^{-1}V_1$$

and,

$$[V_1 \quad V_2] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \quad [V_1 \quad V_2] \tag{1.29}$$

or,

$$V_1A + V_2C = J_2V_1 \tag{1.30}$$

$$V_1B + V_2D + J_2V_2. \tag{1.31}$$

However, from (1.16)

$$C + V_2^{-1}V_1A = DV_2^{-1}V_1 + V_2^{-1}V_1BV_2^{-1}V_1$$

which leaves both (1.30) and (1.31) in the form

$$V_2(D + V_2^{-1}V_1B)V_2^{-1} = J_2 \tag{1.32}$$

verifying the desired spectral decomposition

Corollary 2: The matrix iterations (1.11)-(1.13) and (1.18)-(1.20) are equivalent at every k. Thus, the just proved convergence properties of (1.18)-(1.20) are propagated to (1.11)-(1.13).

Proof: Substitution of (1.17) into (1.18)-(1.20) results in

$$A - BP_{k-1} = A - BP_{k-2} - B(D + P_{k-2}B)^{-1} \cdot R(P_{k-2}) \tag{1.33}$$

$$D + P_{k-1}B = D + P_{k-2}B + (D + P_{k-2}B)^{-1} \cdot R(P_{k-2}) \cdot B \tag{1.34}$$

$$C_k = R(P_{k-1}) = (D + P_{k-2}B)^{-1} \cdot R(P_{k-2}) \cdot (A-BP_{k-1}). \tag{1.35}$$

Letting

$$\alpha_k = A - BP_{k-1} \tag{1.36}$$

$$\gamma_k = D + P_{k-1}B \tag{1.37}$$

$$\sigma_k = C_k \tag{1.38}$$

(1.33) - (1.35) become

$$\alpha_k = \alpha_{k-1} - B\gamma_{k-1}^{-1}\sigma_{k-1} \qquad \alpha_o = A \tag{1.39}$$

$$\gamma_k = \gamma_{k-1} + \gamma_{k-1}^{-1}\sigma_{k-1}B \qquad \gamma_o = D \tag{1.40}$$

$$\sigma_k = \gamma_{k-1}^{-1}\sigma_{k-1}\alpha_k \qquad \sigma_o = C \tag{1.41}$$

which are equivalent to (1.11)-(1.13) $\forall k \geq 0$. This completes the proof.

C. Dual Transformations

Transforming (1.1) into (1.2) can be physically interpreted as removing the "slow" components of the z states. This physical interpretation motivated the Quassi-steady state iterations of the previous section.

The dual to this procedure involves removing the fast parts of the y states. Such a procedure would transform (1.1) into (1.3) satisfying condition (1.5). [6] proposed this dual procedure which lead to matrix recursions

$$A_k = A_{k-1} - B_{k-1}D_{k-1}^{-1}C \qquad A_o = A \tag{1.42}$$

$$B_k = A_kB_{k-1}D_{k-1}^{-1} \qquad B_o = B \tag{1.43}$$

$$D_k = D_{k-1} + CB_{k-1}D_{k-1}^{-1} \qquad D_o = D. \tag{1.44}$$

Likewise, [4,5] have proposed the dual to the Riccati mathod via the transformation

$$\xi = y - \hat{P}z \tag{1.45}$$

which leads to the matrix equations

$$A_k = A - \hat{P}_{k-1}C \qquad A_o = A \tag{1.46}$$

$$D_k = D + C\hat{P}_{k-1} \qquad D_o = D \tag{1.47}$$

$$B_k = S(\hat{P}_{k-1}) = B - \hat{P}_{k-1}D + A\hat{P}_{k-1} - \hat{P}_{k-1}C\hat{P}_{k-1} \qquad B_o = B \tag{1.48}$$

where

$$\hat{P}_{k+1} = \hat{P}_k + S(\hat{P}_k)\cdot(D + C\hat{P}_k)^{-1} \tag{1.49}$$

$$\hat{P}_o = BD^{-1}.$$

We now cite a lemma [3] which is the dual to Lemma 1 and establishes the conditions of convergence of (1.49) to the dichotomic solution.

Lemma 3: If the spectrum of (1.1) satisfies (1.21), then under mild restrictions on $\hat{D}_o$ [3], (1.49) will converge to the dichotomic solution at a convergence rate of (1.22).

Proof: The well known power iteration method [9,10] for computing an M dimensional basis for the dominant right eigenspace of (1.1) is of the form

$$\begin{bmatrix} M_k \\ N_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} M_{k-1} \\ N_{k-1} \end{bmatrix} R_k \qquad \begin{matrix} M_o = B \\ N_o = D \end{matrix} \tag{1.50}$$

where the scaling matrix R_k serves the same purpose as explained in Lemma 1. Thus, under mild restrictions on $\begin{bmatrix} M_o \\ N_o \end{bmatrix}$, it is well known that (1.50) converges to the dominant M-dimensional right eigenspace of (1.1).

Expressing (1.50) as

$$M_k = AM_{k-1} + BN_{k-1} \qquad M_o = B$$

$$N_k^{-1} = N_{k-1}^{-1}(D + CM_{k-1}N_{k-1}^{-1})^{-1} \qquad N_o^{-1} = D^{-1}$$

form the product

$$M_kN_k^{-1} = (B + AM_{k-1}N_{k-1}^{-1})(D + CM_{k-1}N_{k-1}^{-1})^{-1}. \tag{1.51}$$

Letting

$$\hat{P}_k = M_kN_k^{-1},$$

$$\hat{P}_{k+1} = (B + A\hat{P}_k)(D + C\hat{P}_k) \tag{1.52}$$

$$\hat{P}_o = BD^{-1} \tag{1.53}$$

which is equivalent to (1.49) $\forall k \geq 0$. Proving $\hat{P}$ is dichotomic is carried out as in Lemma 1 and can be seen in [3,4].

Corollary 4: The matrix recursions (1.42)-(1.44) and (1.46)-(1.48) are equivalent at every k, thus, the convergence properties of (1.52) are propagated to (1.42)-(1.44).

Proof: Substitution of (1.49) into (1.46)-(1.48) gives

$$A-P_{k-1}C = A - P_{k-2}C - S(P_{k-2})(D + CP_{k-2})^{-1}C \tag{1.54}$$

$$D + CP_{k-1} = D + CP_{k-2} + CS(P_{k-2})(D + CP_{k-2})^{-1} \tag{1.55}$$

$$B_k = S(P_{k-1}) = (A-P_{k-1}C)S(P_{k-2})(D-CP_{k-2})^{-1}. \tag{1.56}$$

Letting

$$\alpha_k = A - P_{k-1}C \tag{1.57}$$

$$\gamma_k = D + CP_{k-1} \tag{1.58}$$

$$\beta_k = B_k \tag{1.59}$$

gives

$$\alpha_k = \alpha_{k-1} - \beta_{k-1}\gamma_{k-1}^{-1}C \qquad \alpha_o = A \tag{1.60}$$

$$\gamma_k = \gamma_{k-1} + C\beta_{k-1}\gamma_{k-1}^{-1} \qquad \gamma_o = D \tag{1.61}$$

$$\beta_k = \alpha_k\beta_{k-1}\gamma_{k-1}^{-1} \qquad \beta_o = B \tag{1.62}$$

which are equivalent to (1.42)-(1.44).

D. Some Comments on the Eigenstructure of a Matrix and its Transpose

Consider the two system

$$\dot{x} = Fx \tag{1.63}$$

$$\dot{\hat{x}} = F^T\hat{x}. \tag{1.64}$$

In the partitioned form of (1.1), (1.63) and (1.64) become

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{1.65}$$

$$\begin{bmatrix} \dot{\hat{y}} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} \hat{y} \\ \hat{z} \end{bmatrix}. \tag{1.66}$$

Now, let us try to decompose time scales in (1.65) and (1.66) independently. First, in (1.65), let

$$w = z + \hat{L}y$$

which transforms (1.65) into

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} (A-B\hat{L}) & B \\ C-D\hat{L}+\hat{L}A-\hat{L}B\hat{L} & D+\hat{L}B \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \tag{1.67}$$

where we seek $\hat{L}$ as the dichotomic solution to

$$R(\hat{L}) = C - D\hat{L} + \hat{L}A - \hat{L}B\hat{L} = 0. \tag{1.68}$$

Now, consider the transformation

$$\hat{x} = \hat{y} - L\hat{z}.$$

applied to (1.66) which gives

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} A^T-LB^T & A^TL+C^T-LB^TL-LD^T \\ B^T & D^T+B^TL \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} \tag{1.69}$$

where we seek L as the dichotomic solution to

$$\hat{R}(L) = C^T - LD^T + A^TL - LB^TL = 0. \tag{1.70}$$

One method of obtaining a solution to (1.70) is through the power iteration

$$\begin{bmatrix} M_{k+1} \\ N_{k+1} \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} M_k \\ N_k \end{bmatrix} \tag{1.71}$$

Letting $L_k = M_k N_k^{-1}$, the dichotomic solution of (1.70) is the equilibrium solution of

$$L_{k+1} = (A^T L_k + C^T)(B^T L_k + D^T)^{-1} \quad (1.72)$$
$$L_o = C^T D^{T^{-1}}.$$

To obtain the dichotomic solution for (1.68) first note that

$$R^T(\hat{L}) = C^T - \hat{L}^T D^T + A^T \hat{L}^T - \hat{L}^T B^T \hat{L}^T = 0. \quad (1.73)$$

Thus, the solution to (1.68) obtained using (1.73) is the transpose of the dichotomic solution to (1.70).

$$\hat{L}_k = (M_k N_k^{-1})T \quad (1.74)$$
$$= N_k^{T^{-1}} M_k^T$$

$$\hat{L}_{k+1} = (D + \hat{L}_k B)^{-1}(L_k A + C) \quad (1.75)$$
$$\hat{L}_o = D^{-1} C$$

and the power iteration (1.71) becomes

$$\begin{bmatrix} M_{k+1}^T \\ N_{k+1}^T \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} M_k^T \\ N_k^T \end{bmatrix} \quad (1.76)$$

or,

$$[M_{k+1} \quad N_{k+1}] = [M_k \quad N_k] \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (1.77)$$

The conclusions are:

i) The power iterations for computing the right dominant eigenspace for (1.66) are equivalent to computing the left dominant digenspace of (1.65).

ii) Removing the "slow" from $\hat{z}$ is equivalent to removing the "fast" from y.

iii) $\hat{L} = L^T$.

iv) The subsystem matrices obtained from the spectrum decomposition are related by

$$(A^T - LB^T) = (A - B\hat{L})^T$$

$$(D^T + B^T L) = (D + \hat{L}B)^T$$

Analogus results hold when the transformations

$$w = z + Ly \ , \ x = y - Lz$$

are applied to (1.65) and (1.66) respectively.

E. Block Diagonalization and Identification of Fast and Slow State Vector Components

Once we have transformed (1.1) into (1.2) or (1.3) satisfying conditions (1.4) or (1.5) respectively, block diagonalization is always possible.

Consider form (1.2), and the transformation (1.14) used to obtain this form. The dichotomic solution matrix P is of the form

$$P = N^{-1}M$$

where the rows of [M N] span the dominant left eigenspace of (1.1). Thus, the exact form of (2) is

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - BP & B \\ 0 & D + PB \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \tag{1.78}$$

Now, let $x = y - Qw$. This leaves (1.78) in the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - BP & (A - BP)Q - Q(D + PB) + B \\ 0 & D + PB \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{1.79}$$

Thus, we seek Q to satisfy the Lyapunov type equation

$$(A - BP)Q - Q(D + PB) + B = 0 \tag{1.80}$$

Such a Q will always exist since

$$\sigma(A - BP) \cap \sigma(D + PB) = \emptyset \tag{1.81}$$

(1.80) may be solved algebraically [12] or iteratively [4]. One obvious iterative scheme is to apply the dominant right eigenspace iterations used for transforming (1.1) into (1.3). Since (1.78) satisfies (1.21) convergence is assured. Such an iteration would take the form

$$Q_{k+1} = (B + (A - BP)Q_k) \cdot (D + PB)^{-1}$$

$$Q_o = 0$$

whichever method used, the resulting system is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - BP & 0 \\ 0 & D + PB \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{1.82}$$

and the composite transformation is

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I & Q \\ -P & I - PQ \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{1.83}$$

which possesses the explicit inverse

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} I - QP & -Q \\ P & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{1.84}$$

Thus, we have decomposed the y and z state vectors into their respective "fast" and "slow" components. Namely

$$y = x + Qw = y_{slow} + y_{fast} \tag{1.85}$$

$$z = -Px + (I - PQ)w = z_{slow} + z_{fast} \tag{1.86}$$

where

$$x(t) = e^{(A - BP)t} x_o \qquad x_o = (I - QP)y_o - Qz_o$$

$$w(t) = e^{(D + PB)t} w_o \qquad w_o = Py_o + z_o$$

Such decompositions will become more important when we consider singularly perturbed systems in the next chapter. There, the fast and slow components take on the names of "Boundary Layer" and "Steady State Components".

Now consider form (1.3) and the transformation (1.45) used to obtain this form. The Dichotomic solution matrix $\hat{P}$ is of the form

$$\hat{P} = MN^{-1} \tag{1.87}$$

Where the columns of $[M\ N]^T$ span the dominant right eigenspace of (1). Thus, the exact form of (1.3) is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - PC & 0 \\ C & D + CP \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{1.88}$$

Now, let $w = z + Qx$. This transforms (1.88) into the form

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} A - PC & 0 \\ Q(A - PC) - (D + CP)Q + C & D + CP \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{1.89}$$

Thus we seek Q to satisfy the Lyapunov type equation

$$Q(A - PC) - (D + CD)Q + C = 0 \tag{1.90}$$

such a Q will always exist since

$$\sigma(A - DC) \cap \sigma(D + CP) = \phi \tag{1.91}$$

Again, (1.90) may be solved iteratively or algebraically. Applying the dominant left eigenspace iterations to (1.88) convergence is assured. This iteration takes the form

$$Q_{k+1} = (D + CP)^{-1} \cdot (C + Q_k \cdot (A - PC)) \qquad Q_o = D^{-1}C \tag{1.92}$$

The resulting system is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - PC & 0 \\ 0 & D + CP \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{1.93}$$

and the composite transformation is

$$\begin{bmatrix} \dot{x} \\ w \end{bmatrix} = \begin{bmatrix} I & -P \\ Q & I - QP \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{1.94}$$

with also possesses the explicit inverse

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I - PQ & P \\ -Q & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{1.95}$$

Thus, we have again decomposed the y and z state vectors into "Fast" and "Slow" components. Namely

$$y = (I - PQ)x + Pw = y_{slow} + y_{fast} \tag{1.96}$$

$$z = -Qx + w = z_{slow} + z_{fast} \tag{1.97}$$

where

$$x(t) = e^{(A - PC)}x_o \qquad x_o = y_o - Pz_o$$

$$w(t) = e^{(D + CP)}w_o \qquad w_o = Qy_o + (I - QP)z_o$$

The relationships between various fast and slow components of (1.1) will be made apparant in the next chapter.

F. Ordering of States Variables

In both the dominant left and right eigenspace iterations, it was assumed that if (1.21) held, then

$$(D + P_k B)^{-1} \tag{1.98}$$

$$(D + C\hat{P}_k)^{-1} \tag{1.99}$$

exists for every k. (1.17) is used to obtain the dichotomic solution of (1.16). This solution was shown to be of the form

$$P = V_{22}^{-1} V_{21} \tag{1.100}$$

Where $[V_{21} \; V_{22}]$ are left eigenvectors corresponding to the dominant eigenvectors of (1.1). If V_{22}^{-1} does not exist, then neither will (1.100). To analyse this problem look at

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

or,

$$V_{11}A_{11} + V_{12}A_{21} = \Lambda_1 V_{11} \tag{1.101}$$

$$V_{11}A_{12} + V_{12}A_{22} = \Lambda_1 V_{12} \tag{1.102}$$

$$V_{21}A_{11} + V_{22}A_{21} = \Lambda_2 V_{21} \tag{1.103}$$

$$V_{21}A_{12} + V_{22}A_{22} = \Lambda_2 V_{22} \tag{1.104}$$

[14] has shown that for arbitrary N, M, there will exist an ordering of the states (eigenvectors) such that V_{22}^{-1} exists. However, [14] does not guarantee that the resulting spectrums of Λ_1 and Λ_2 satisfy (1.21).

In the next chapter we will consider singularly perturbed systems. In this case, the system matrix possesses certain explicit properties that make it well suited for (1.98) and (1.99). To see this, assume

$$\|A_{22}\| = \|A_{21}\| = \|\Lambda_2\| = O(1/\mu) \tag{1.105}$$

Then (1.104) can be written as

$$V_{22}A_{22} + O(\mu) = \Lambda_2 V_{22} \tag{1.106}$$

Thus, V_{22} is $O(\mu)$ close to being the left eigenvector matrix for A_{22}. Using the relations (1.105), (1.98) becomes

$$(D + \mu P_k B)^{-1} \tag{1.107}$$

Using bounds on μ established by [4] on P, we can establish sufficient bounds on μ guaranteeing the existence of (1.108) $\forall_k$, thus guaranteeing the existence of V_{22}^{-1}. Thus, the ordering of states such as to fit a singularly perturbed format is very important to the success of (1.97) and (1.98). [3] gives an algorithm for putting an arbitrary system into singularly perturbed format and should be used before any multiple time scale analysis is attempted.

Likewise, to analyse (1.98),

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{1.108}$$

or,

$$A_{11}E_{11} + A_{12}E_{21} = E_{11}\Lambda_1 \tag{1.109}$$

$$A_{11}E_{12} + A_{12}E_{22} = E_{12}\Lambda_2 \tag{1.110}$$

$$A_{21}E_{11} + A_{22}E_{21} = E_{21}\Lambda_1 \tag{1.111}$$

$$A_{21}E_{12} + A_{22}E_{22} = E_{22}\Lambda_2 \tag{1.112}$$

using (1.105), (1.112) and (1.110) yield

$$O(\mu) + A_{22}E_{22} = E_{22}\Lambda_2 \tag{1.113}$$

and again the desired properties become obvious.

It is interesting to note that if E_{11}^{-1} and V_{11}^{-1} exist, then using

$$VE = I, \tag{1.114}$$

we obtain

$$V_{22}^{-1}V_{21} = -E_{21}E_{11}^{-1} \tag{1.115}$$

$$E_{12}E_{22}^{-1} = -V_{11}^{-1}V_{12} \tag{1.116}$$

The quantities on the left of (1.15) and (1.116) are the equilibruim solutions to the dominant left and right eigenspace iterations respectively. Thus the fact that we ordered our states so that there were M dominant eigenvalues and the z state, were fast was arbitrary in that by reordering our states, the right eigenspace method may be used for (1.2) and the left for (1.3). The existence of all of these inverses simulataneously is unlikely. However, the relations (1.115) and (1.116) do help to further clarify the dual nature of our left and right eigenspace transformations.

G. Example - Decomposition of States into Fast and Slow Components

In [13], the 8th order model of an isolated mixed power system is given as

$$\dot{x} = \begin{bmatrix} -.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.75 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .16667 & -.16667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & -.08 & -.07467 & -.112 & -3.9944 & 10 & -.92778 & -9.1 \\ 0 & 0 & 0 & 0 & .2 & -.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.39 & -.278 \\ 0 & .01 & .0093 & .014 & -.06319 & 0 & .11597 & -.112361 \end{bmatrix} x \tag{1.117}$$

using a permutation of $x = Py$

$$P = (e_1, e_3, e_6, e_8, e_5, e_4, e_7, e_2)$$

gives

$$\dot{y} = \begin{bmatrix} -.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.167 & 0 & 0 & 0 & 0 & 0 & .167 \\ 0 & 0 & -.5 & 0 & .2 & 0 & 0 & 0 \\ 0 & .009 & 0 & -.112 & -.063 & .014 & .116 & 01 \\ 0 & -.075 & 10.00 & -9.101 & -3.994 & -.112 & -.927 & -.08 \\ 0 & 2.00 & 0 & 0 & 0 & -2.00 & 0 & 0 \\ 0 & 0 & 0 & -.277 & 1.319 & 0 & -1.386 & 0 \\ 4.75 & 0 & 0 & 0 & 0 & 0 & 0 & -5.00 \end{bmatrix} y \tag{1.118}$$

The eigenvalues of (1.118) are

$$\begin{array}{rcl} -1.3884147 & + & 0.0000000J \\ -0.1291288 & + & 0.2124795J \\ -0.1291288 & - & 0.2124795J \\ -4.3489879 & + & 0.0000000J \\ -2.0000000 & + & 0.0000000J \\ -0.1666700 & + & 0.0000000J \\ -5.0000000 & + & 0.0000000J \\ -0.2000000 & + & 0.0000000J \end{array} \tag{1.119}$$

Using $N = M = 4$, we obtain an ϵ of .1792. Using the dominant left eigenspace iterations we obtain

$$A - P_oB = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.15834 & -0.16667 & 0.00000 & 0.00000 \\ -0.00312 & -0.00766 & -0.08981 & -0.36571 \\ 0.00877 & 0.02153 & 0.09635 & -0.22145 \end{bmatrix} \tag{1.120}$$

which has eigenvalues

$$\begin{matrix} -0.15563 & + & 0.17579J \\ -0.15563 & - & 0.17579J \\ -0.16667 & + & 0.00000J \\ -0.20000 & + & 0.00000J \end{matrix} \tag{1.121}$$

$$D + BP_o = \begin{bmatrix} -4.52014 & -0.08640 & -0.71572 & -0.05533 \\ 0.00000 & -2.00000 & 0.00000 & -0.16667 \\ 0.80733 & 0.02712 & -1.16426 & 0.02543 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{1.122}$$

which has eigenvalues

$$\begin{matrix} -4.33808 & + & 0.00000J \\ -1.34632 & + & 0.00000J \\ -2.00000 & + & 0.00000J \\ -5.00000 & + & 0.00000J \end{matrix} \tag{1.123}$$

using the dominant right eigenspace iterations we get

$$A - C\hat{P}_o = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.15834 & -0.16667 & 0.00000 & 0.00000 \\ -0.00312 & -0.00766 & -0.08981 & -0.36571 \\ 0.00877 & 0.02153 & 0.09635 & -0.22145 \end{bmatrix} \tag{1.124}$$

which has eigenvalues

$$\begin{matrix} -0.15563 & + & 0.17579J \\ -0.15563 & - & 0.17579J \\ -0.16667 & + & 0.00000J \\ -0.20000 & + & 0.00000J \end{matrix} \tag{1.125}$$

$$D + \hat{P}_o C = \begin{bmatrix} -4.31691 & -0.03023 & 0.04757 & -0.05415 \\ 0.00000 & -2.00000 & 0.00000 & -0.06667 \\ 1.32208 & 0.00179 & -1.36749 & 0.00051 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{1.126}$$

Which has eigenvalues

$$\begin{matrix} -4.33808 & + & 0.00000J \\ -1.34632 & + & 0.00000J \\ -2.00000 & + & 0.00000J \\ -5.00000 & + & 0.00000J \end{matrix} \tag{1.127}$$

To show how this accuracy may be improved, after two iterations we obtain

$$A - PB = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.16492 & -0.16667 & 0.00000 & 0.00000 \\ -0.00233 & -0.00741 & -0.08420 & -0.36234 \\ 0.00979 & 0.02836 & 0.13210 & -0.17633 \end{bmatrix} \tag{1.128}$$

with eigenvalues

$$\begin{matrix} -0.13027 & + & 0.21388J \\ -0.13027 & - & 0.21388J \\ -0.16667 & + & 0.00000J \\ -0.20000 & + & 0.00000J \end{matrix} \tag{1.129}$$

$$D + BP = \begin{bmatrix} -4.52468 & -0.08664 & -0.71767 & -0.05571 \\ 0.00000 & -2.00000 & 0.00000 & -0.18172 \\ 0.76779 & 0.02154 & -1.21045 & 0.01334 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{1.130}$$

with eigenvalues

$$\begin{matrix} -4.34912 & + & 0.00000J \\ -1.38601 & + & 0.00000J \\ -2.00000 & + & 0.00000J \\ -5.00000 & + & 0.00000J \end{matrix} \tag{1.131}$$

$$A - C\hat{P} = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.16379 & -0.16667 & 0.00000 & 0.00000 \\ -0.00237 & -0.00609 & 0.00444 & -0.45814 \\ 0.00821 & 0.02283 & 0.13946 & -0.26497 \end{bmatrix} \tag{1.132}$$

with eigenvalues

$$\begin{array}{rcl} -0.13027 & + & 0.21388J \\ -0.13027 & - & 0.21388J \\ -0.16667 & + & 0.00000J \\ -0.20000 & + & 0.00000J \end{array} \tag{1.133}$$

$$D + \hat{P}C = \begin{bmatrix} -4.37192 & -0.03420 & -0.05145 & -0.05565 \\ 0.00000 & -2.00000 & 0.00000 & -0.06896 \\ 1.32327 & 0.00202 & -1.36321 & 0.00048 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \tag{1.134}$$

with eigenvalues

$$\begin{array}{rcl} -4.34912 & + & 0.00000J \\ -1.38601 & + & 0.00000J \\ -2.00000 & + & 0.00000J \\ -5.00000 & + & 0.00000J \end{array} \tag{1.135}$$

We now give graph of the various states along with their fast and slow components using (1.85), (1.86), (1.96) and (1.97) for both the left and right eigenspace decompositions. The plots will be based on the P_o and $\hat{P}_o$ iterates. On the graphs of the individual components, the following legend will be in effect

ACTUAL STATE ————————
SLOW COMPONENT ----------------
FAST COMPONENT ················

On the graphs of the actual state versus the approximated state

ACTUAL STATE ————————
APPROXIMATED STATE ···············

The plots appear on the next several pages. The system is perturbed with an initial state vector of

with an initial state vector of

$$X_o^T = (1,\ 2,\ 3,\ -2,\ 1,\ -1,\ 4,\ 2) \tag{1.136}$$

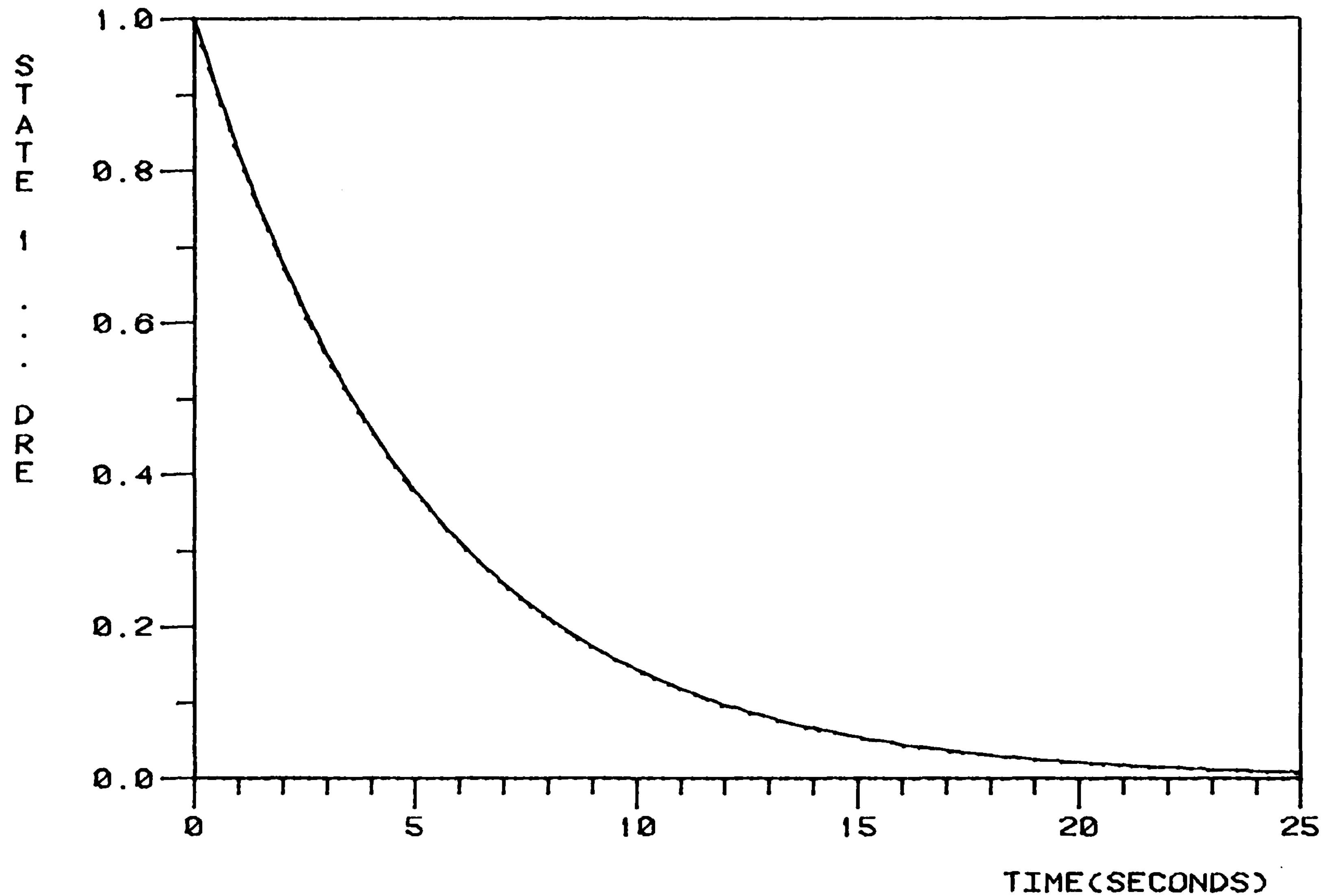

Figure 1.1. State 1 and components using eight eigenspace iterations.

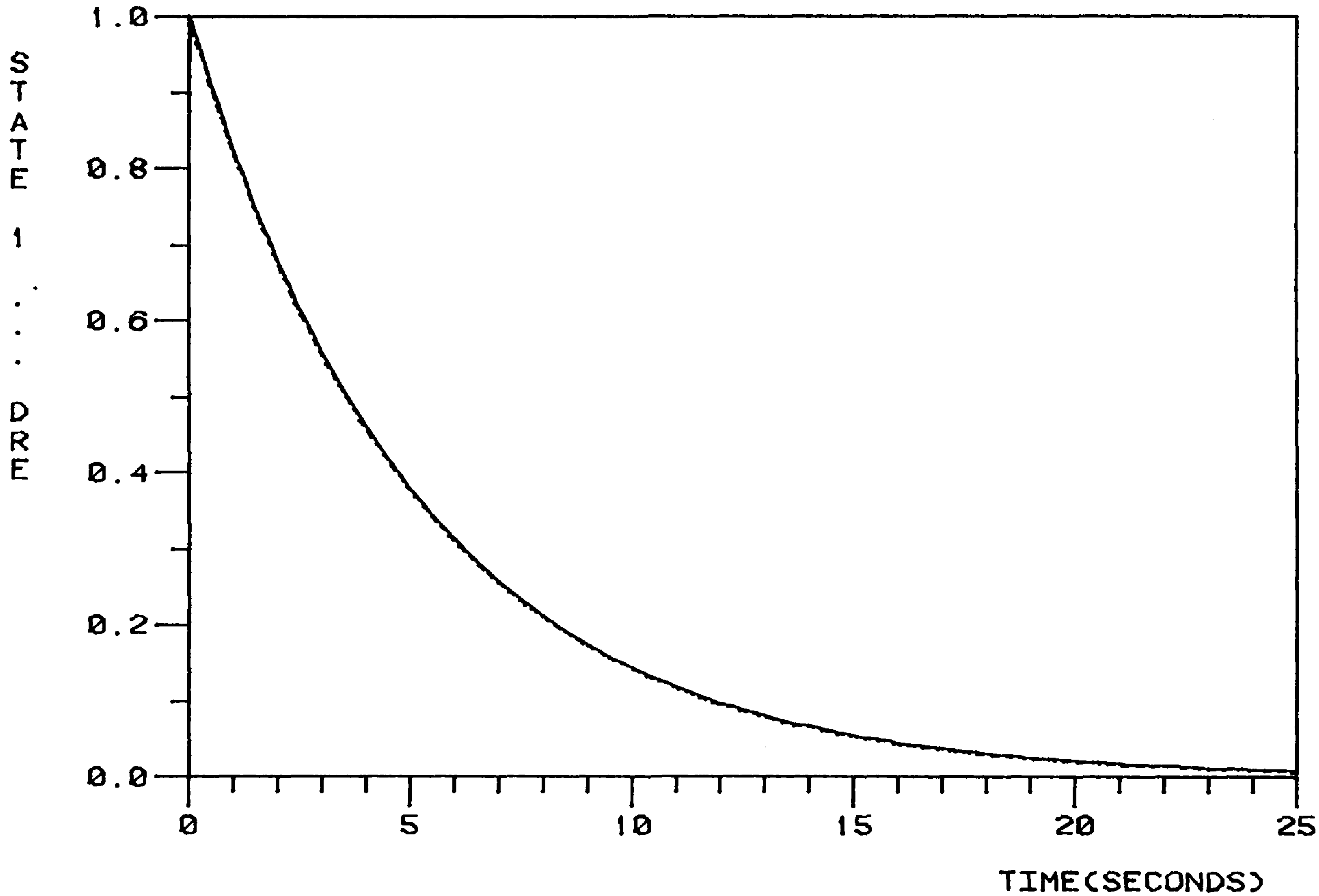

Figure 1.2. State 1 and added components using right eigenspace iterations.

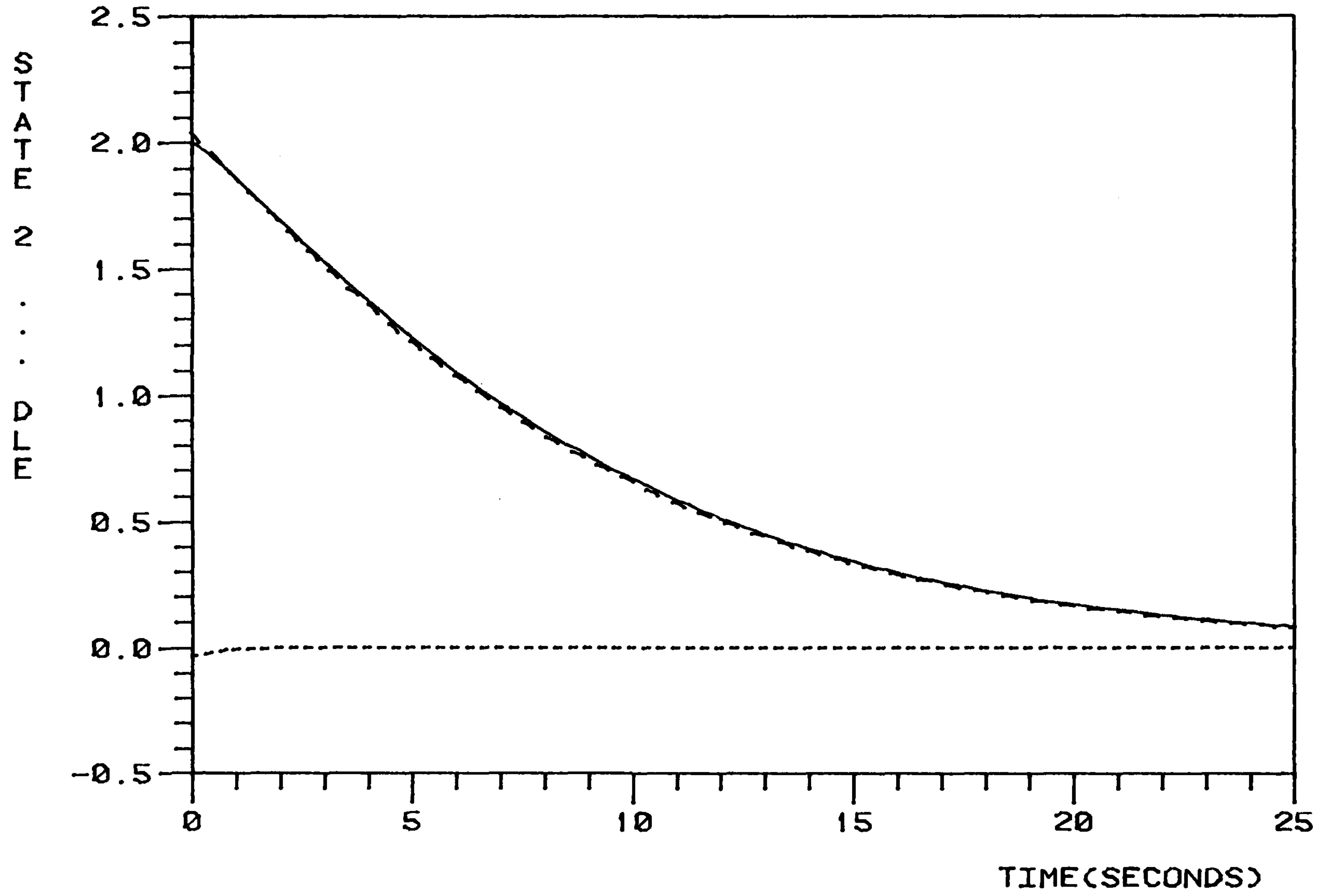

Figure 1.3. State 2 and components using left eigenspace iterations.

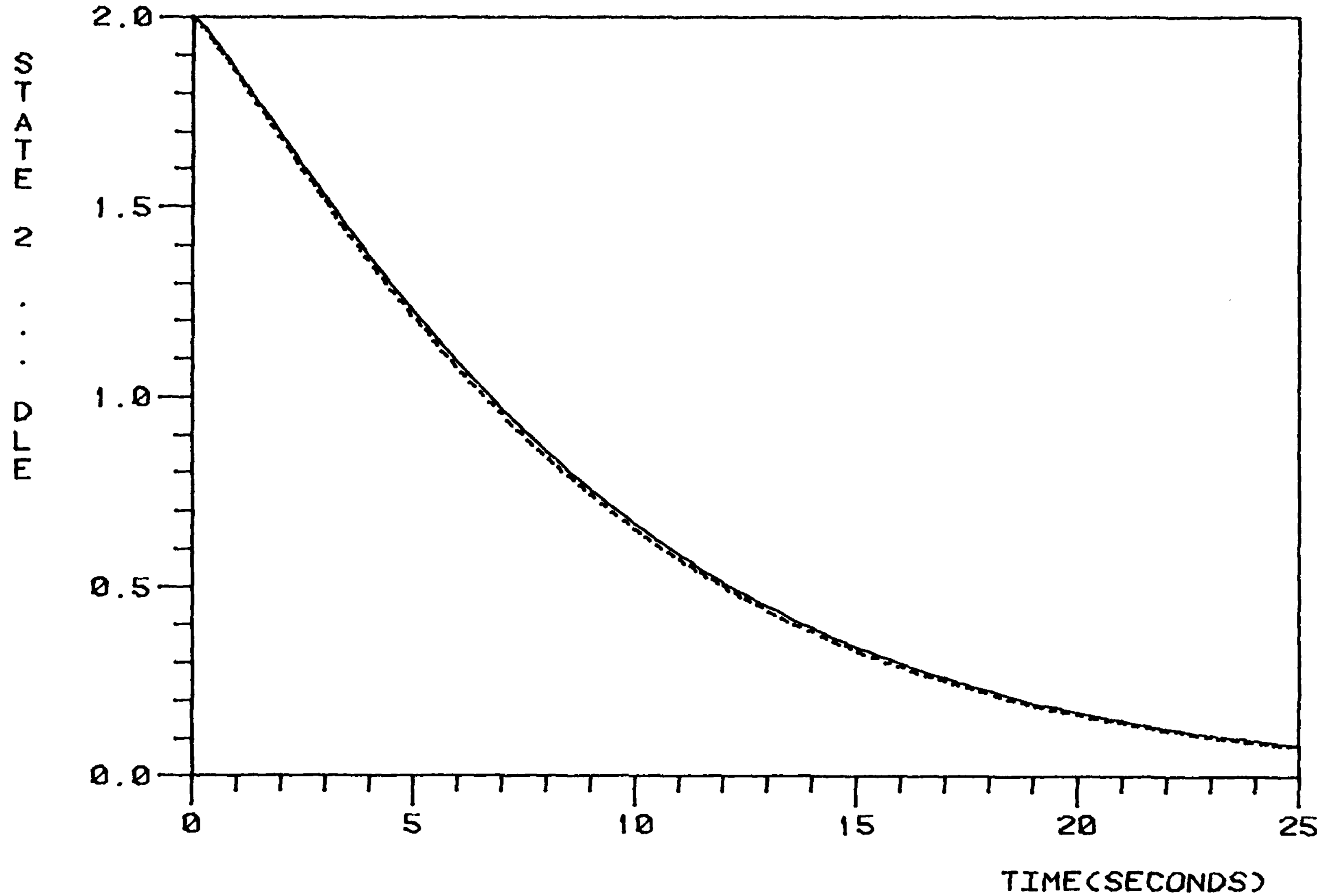

Figure 1.4. State 2 and added components using left eigenspace iterations.

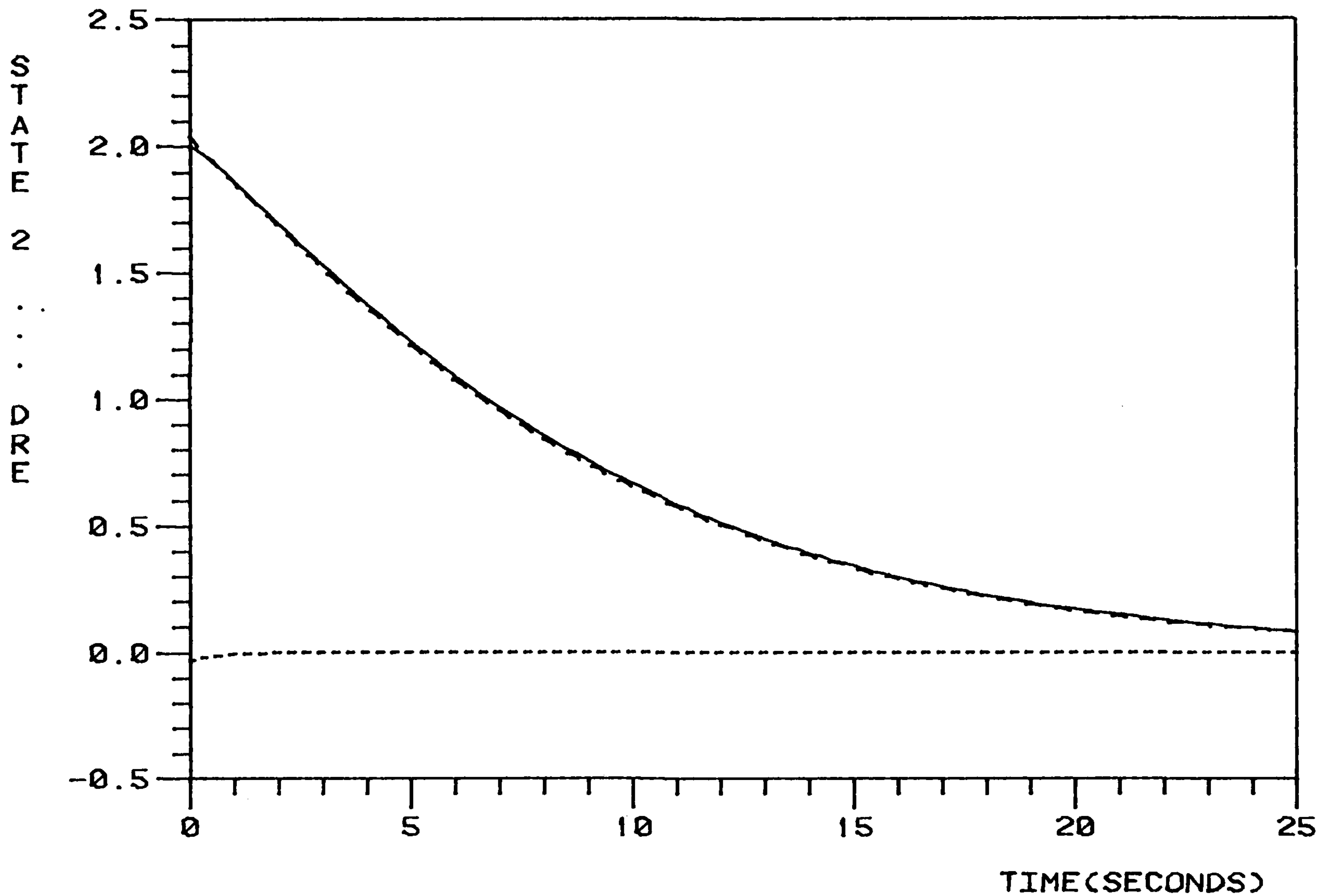

Figure 1.5. State 2 and components using right eigenspace iterations.

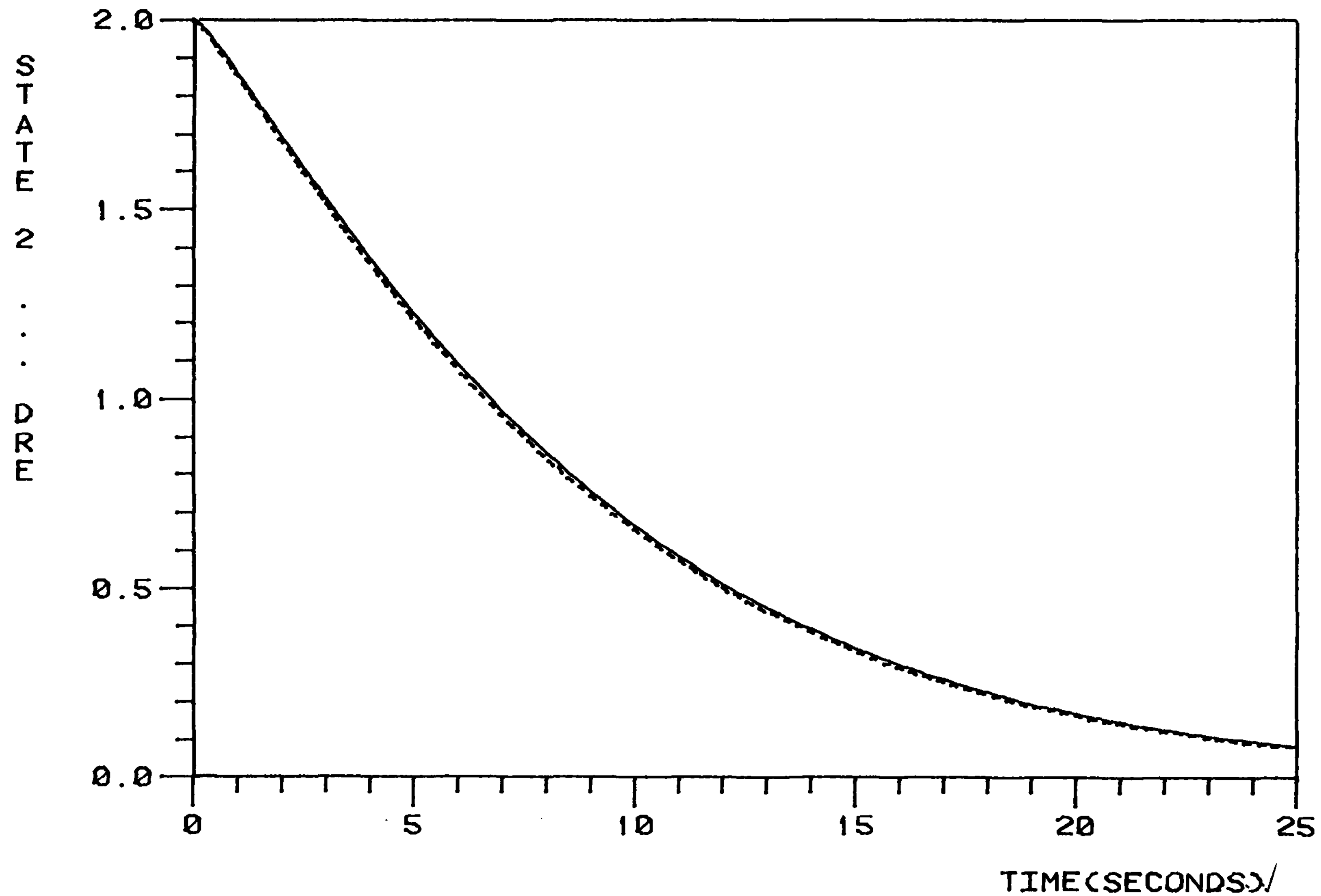

Figure 1.6. State 2 and added components using right eigenspace iterations.

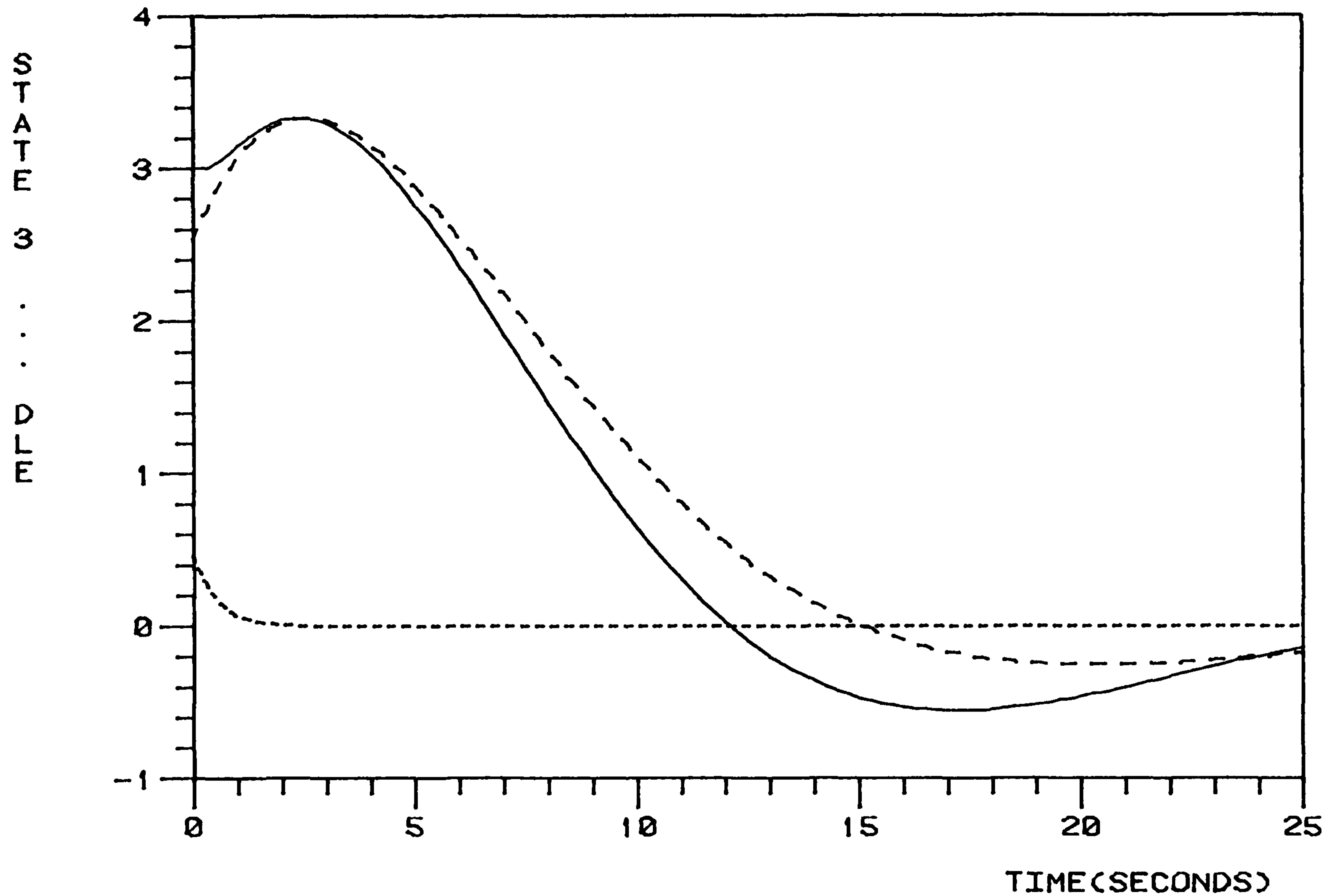

Figure 1.7. State 3 and components using left eigenspace iterations.

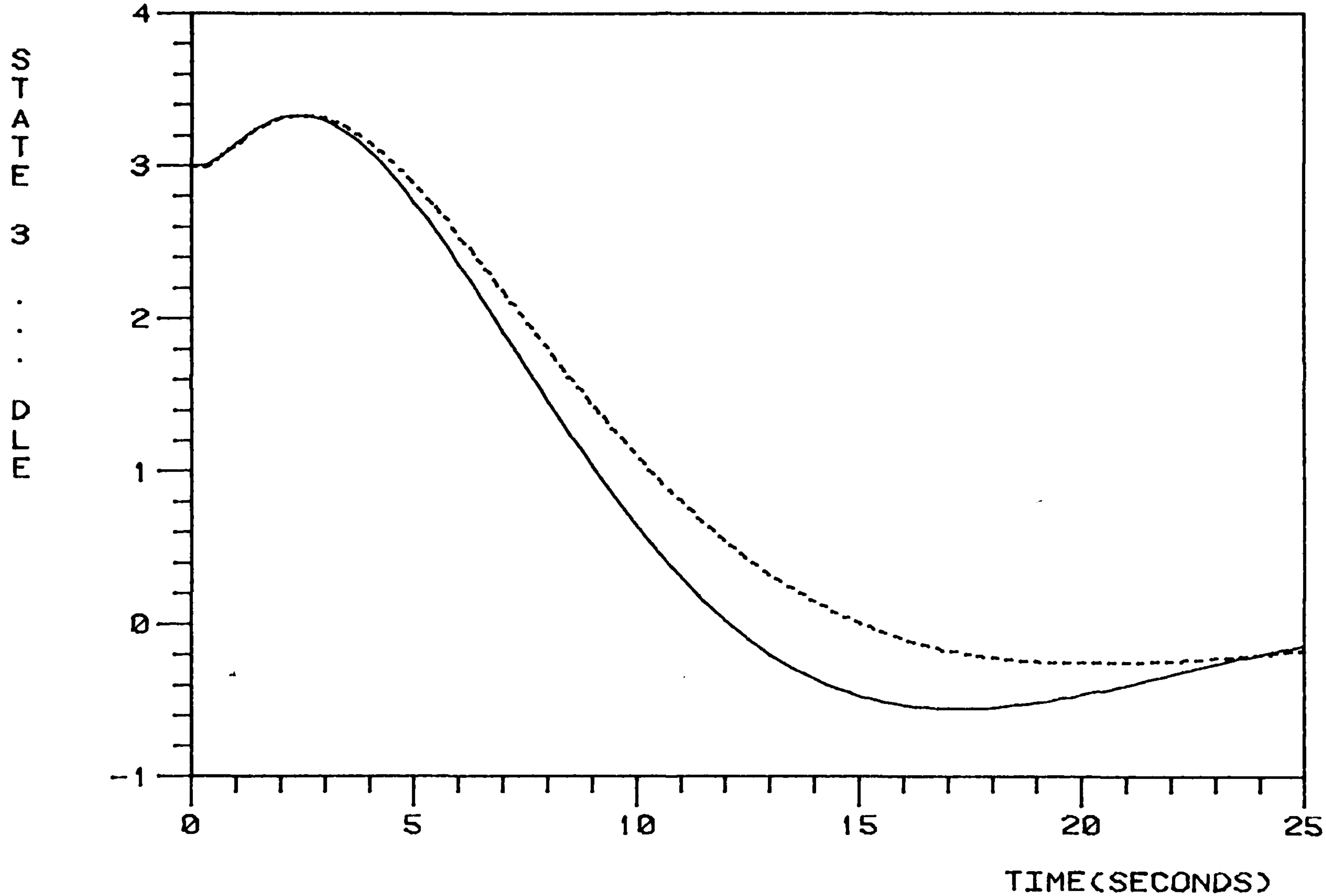

Figure 1.8. State 3 and added components using left eigenspace iterations.

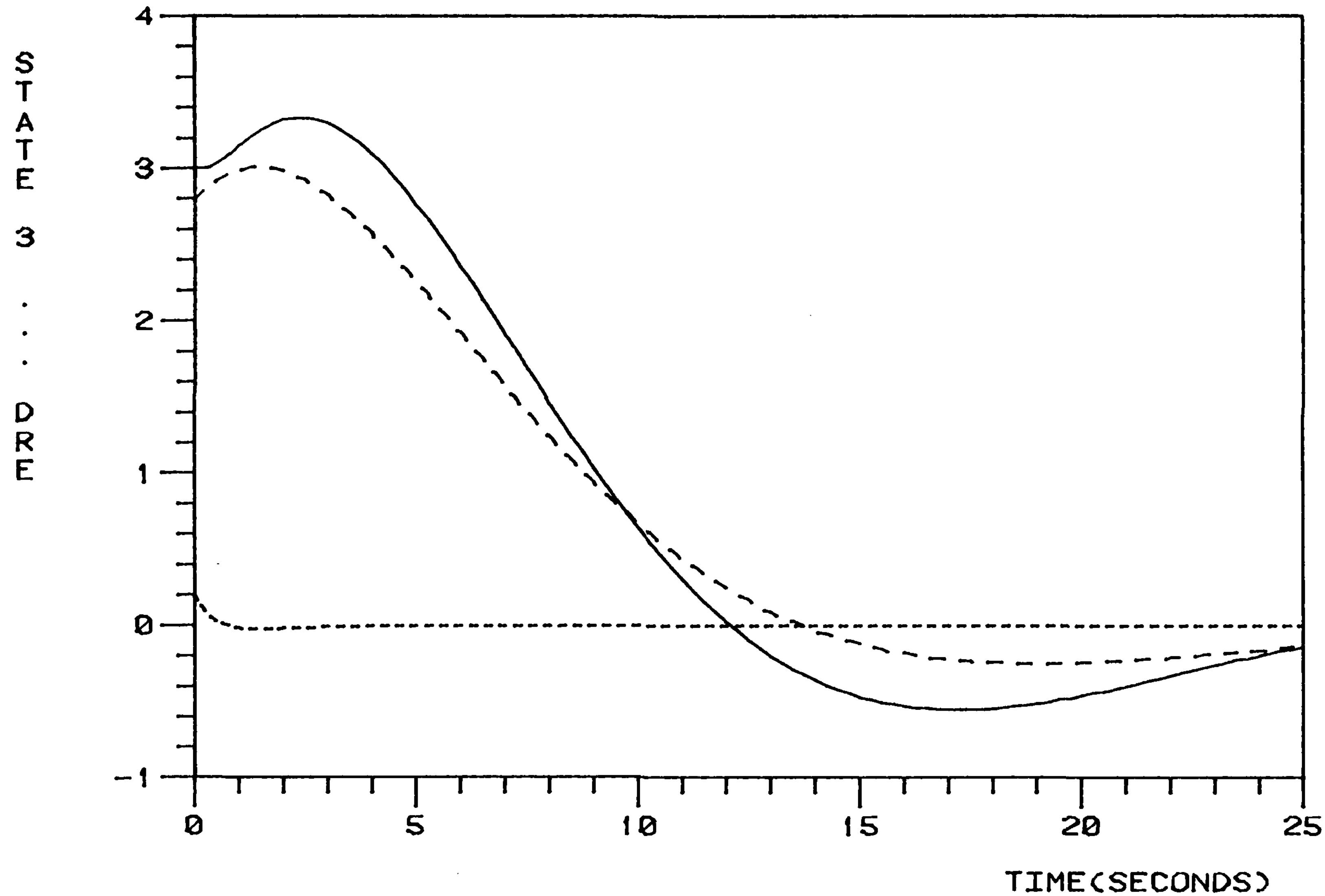

Figure 1.9. State 3 and components using right eigenspace iterations.

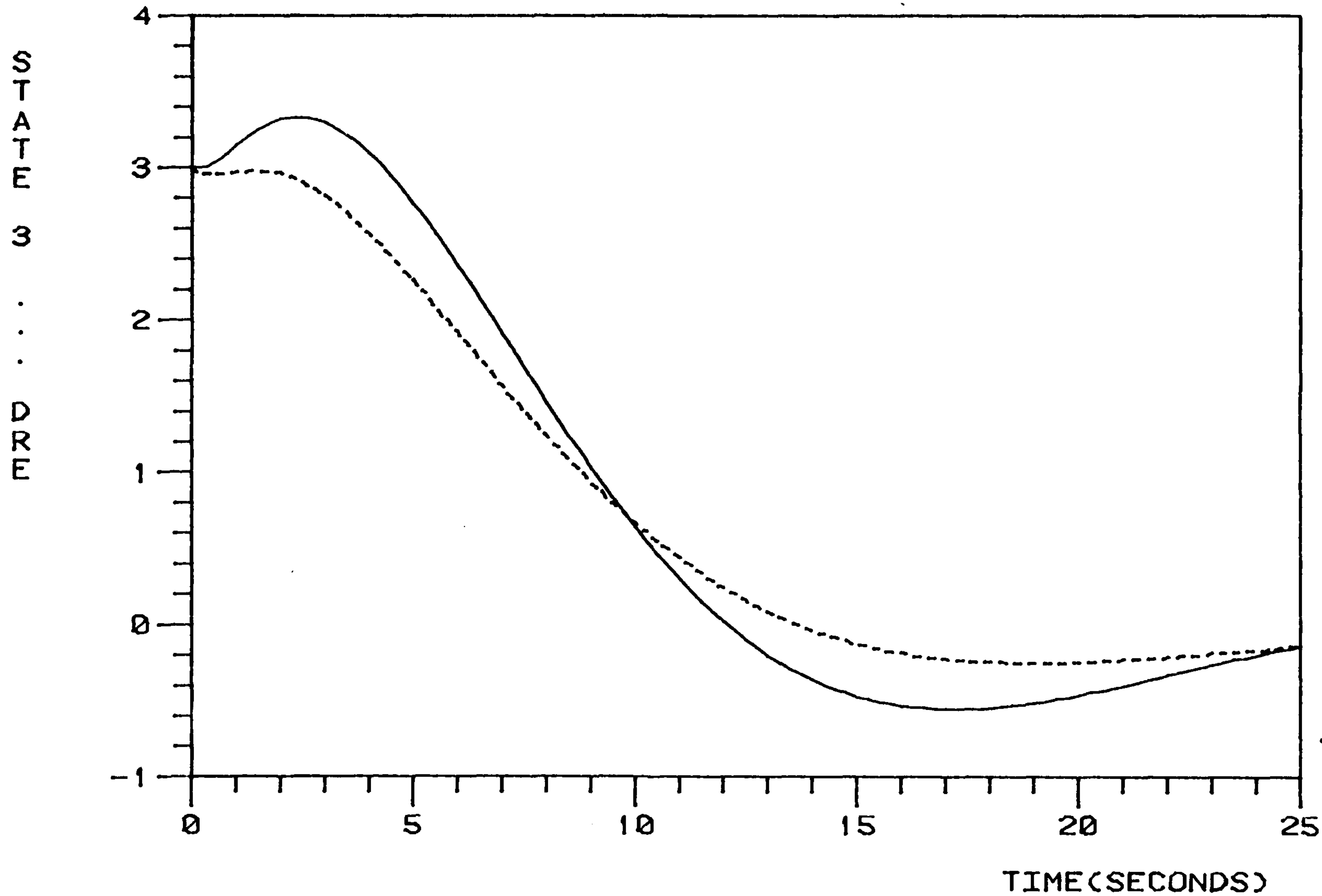

Figure 1.10. State 3 and added components using right eigenspace iterations.

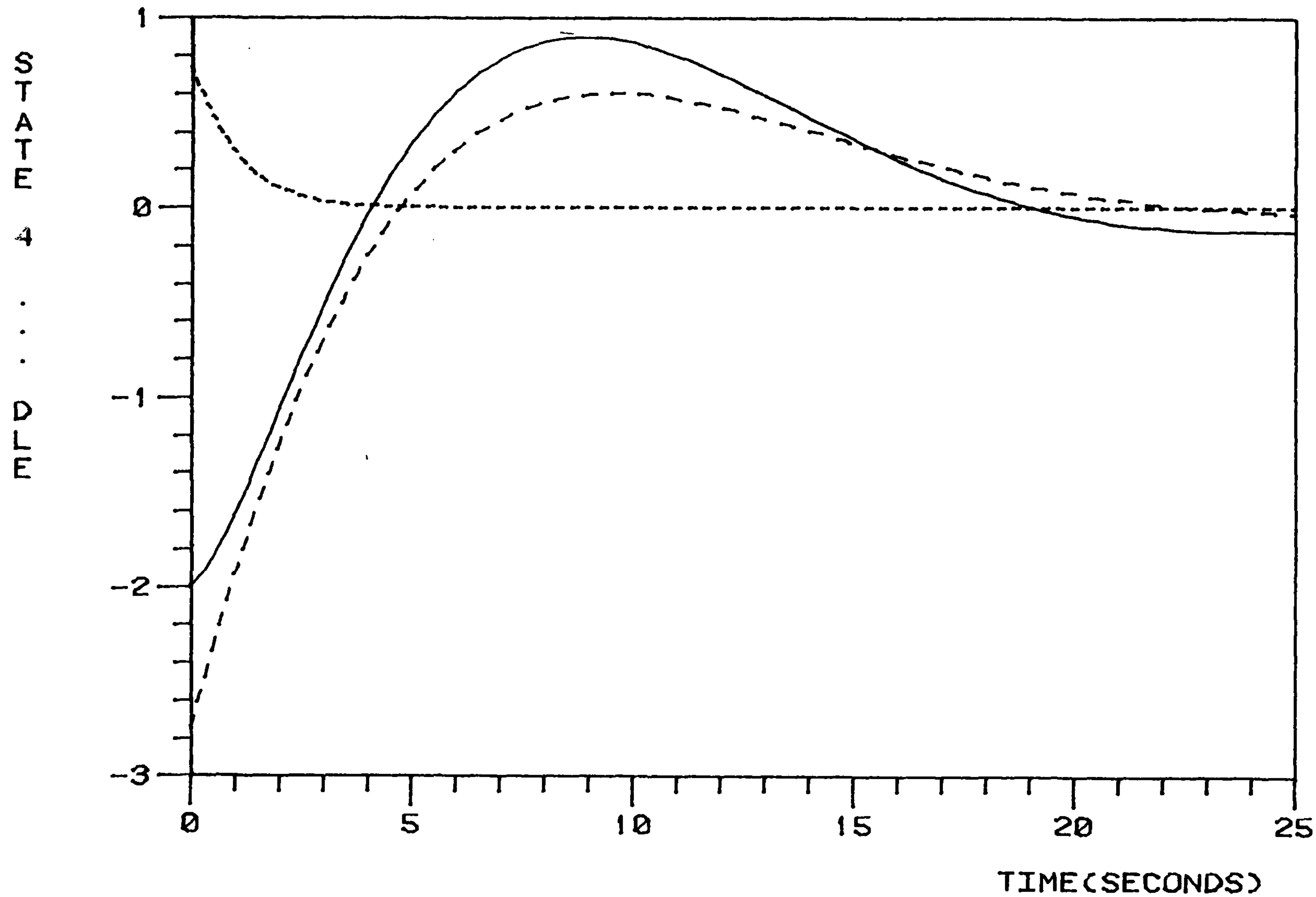

Figure 1.11. State 4 and components using left eigenspace iterations.

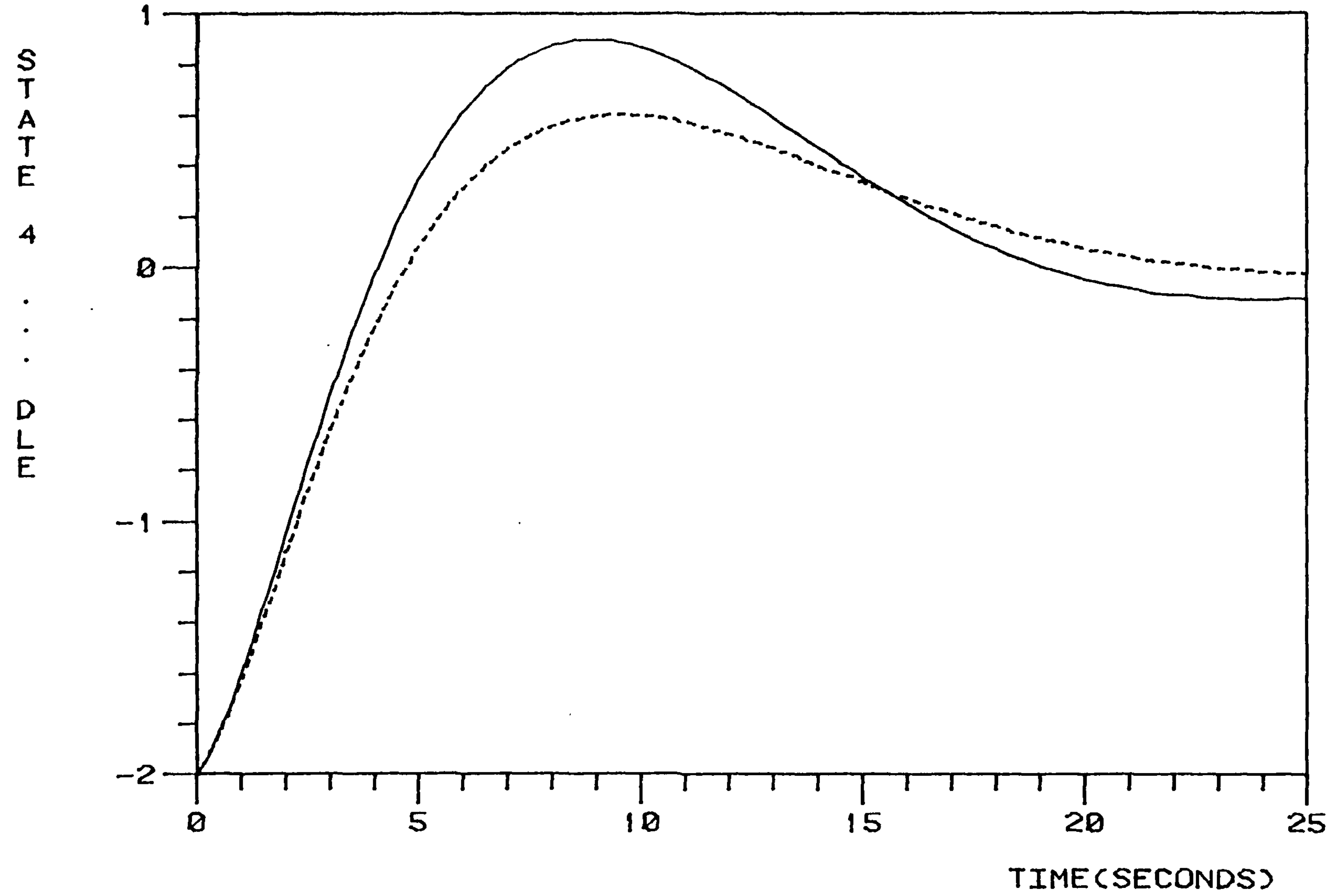

Figure 1.12. State 4 and added components using left eigenspace iterations.

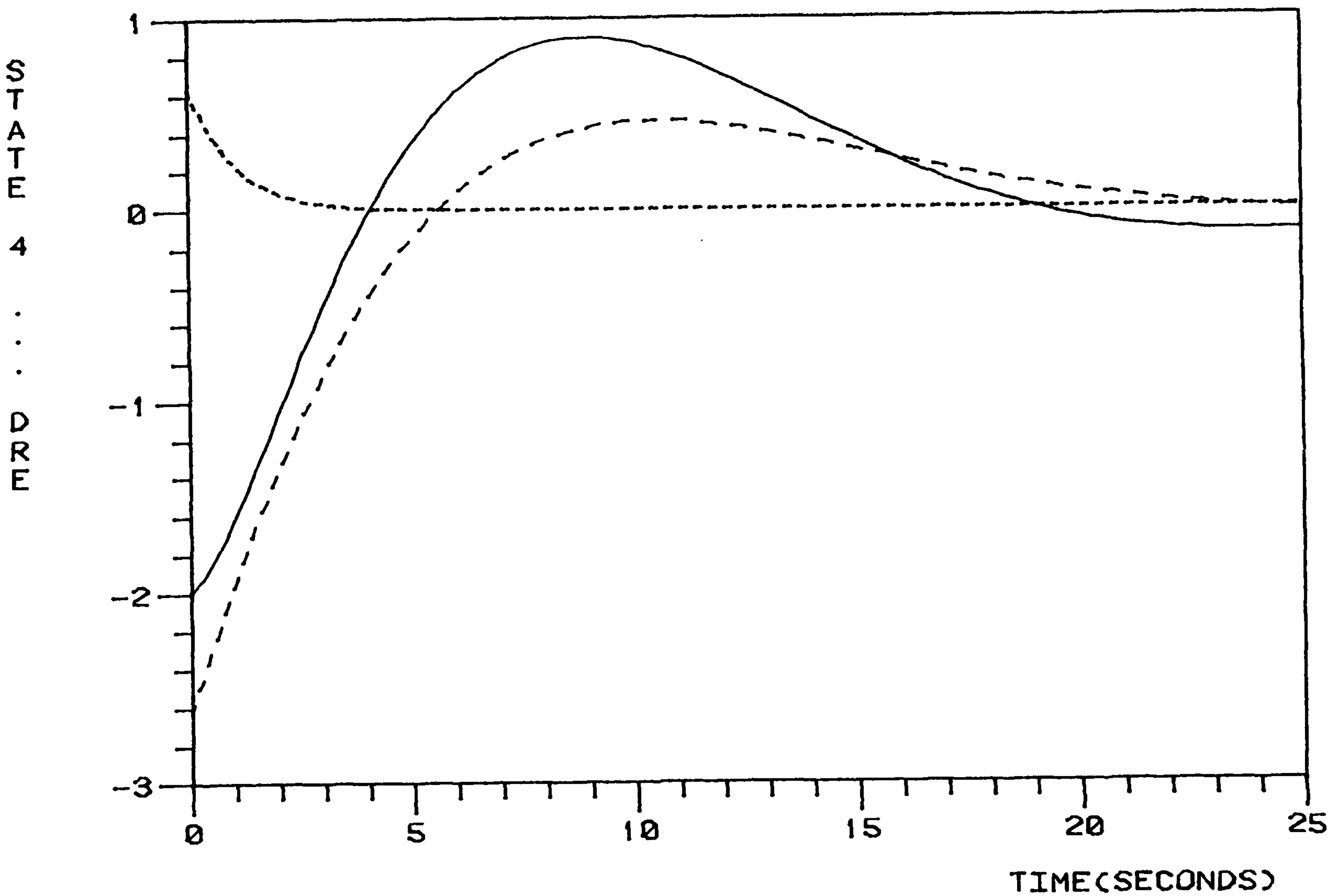

Figure 1.13. State 4 and components using right eigenspace iterations.

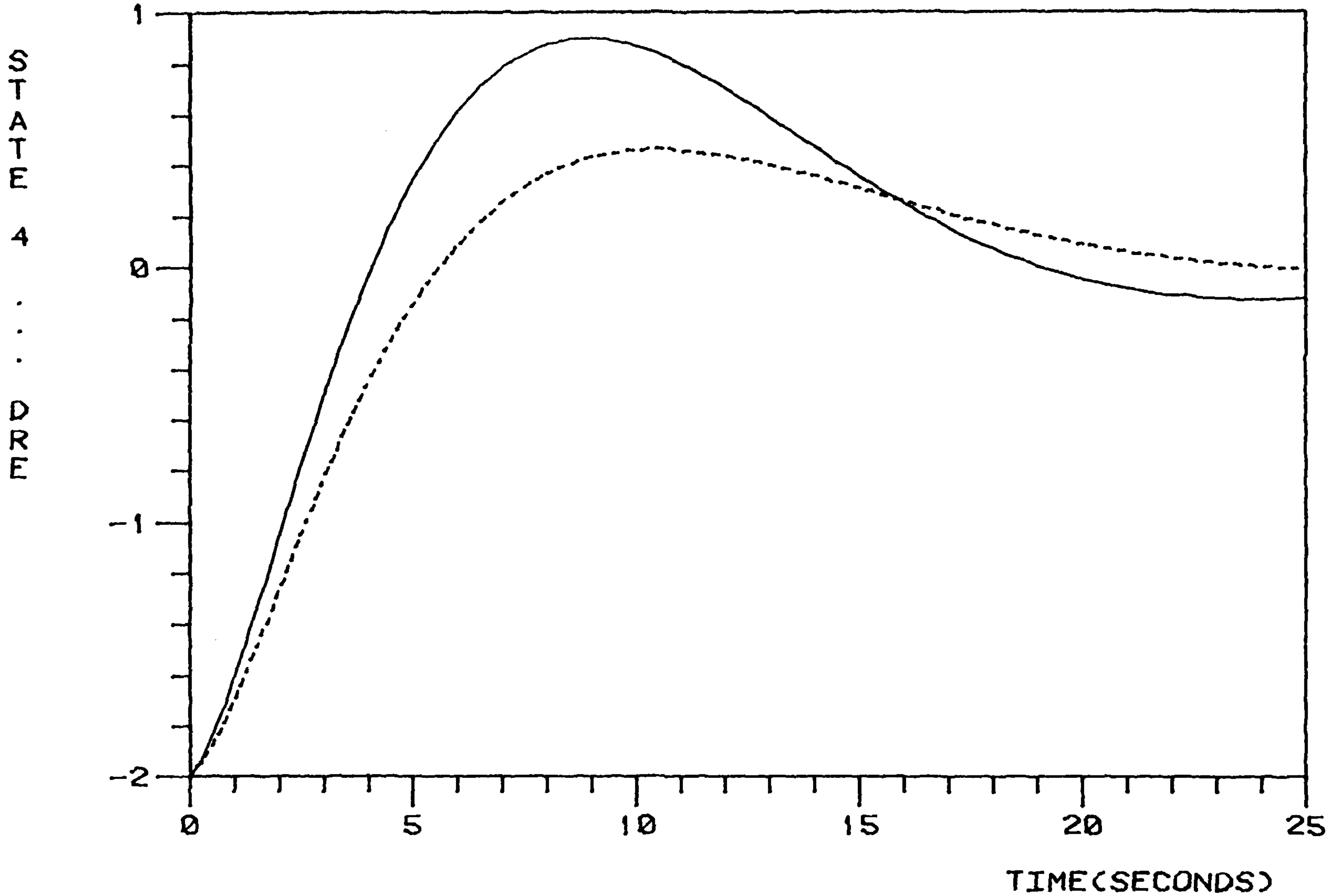

Figure 1.14. State 4 and added components using right eigenspace iterations.

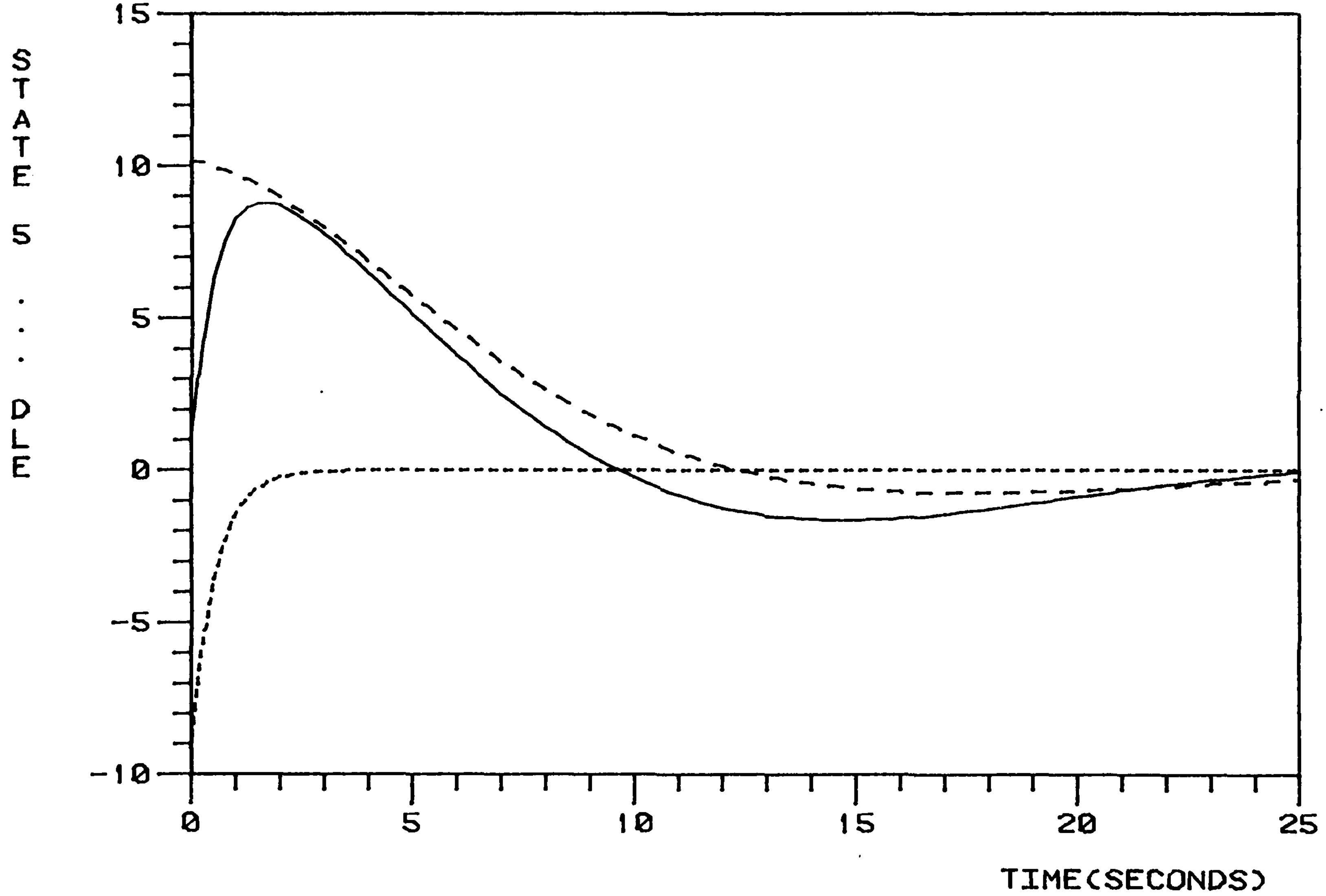

Figure 1.15. State 5 and components using left eigenspace iterations.

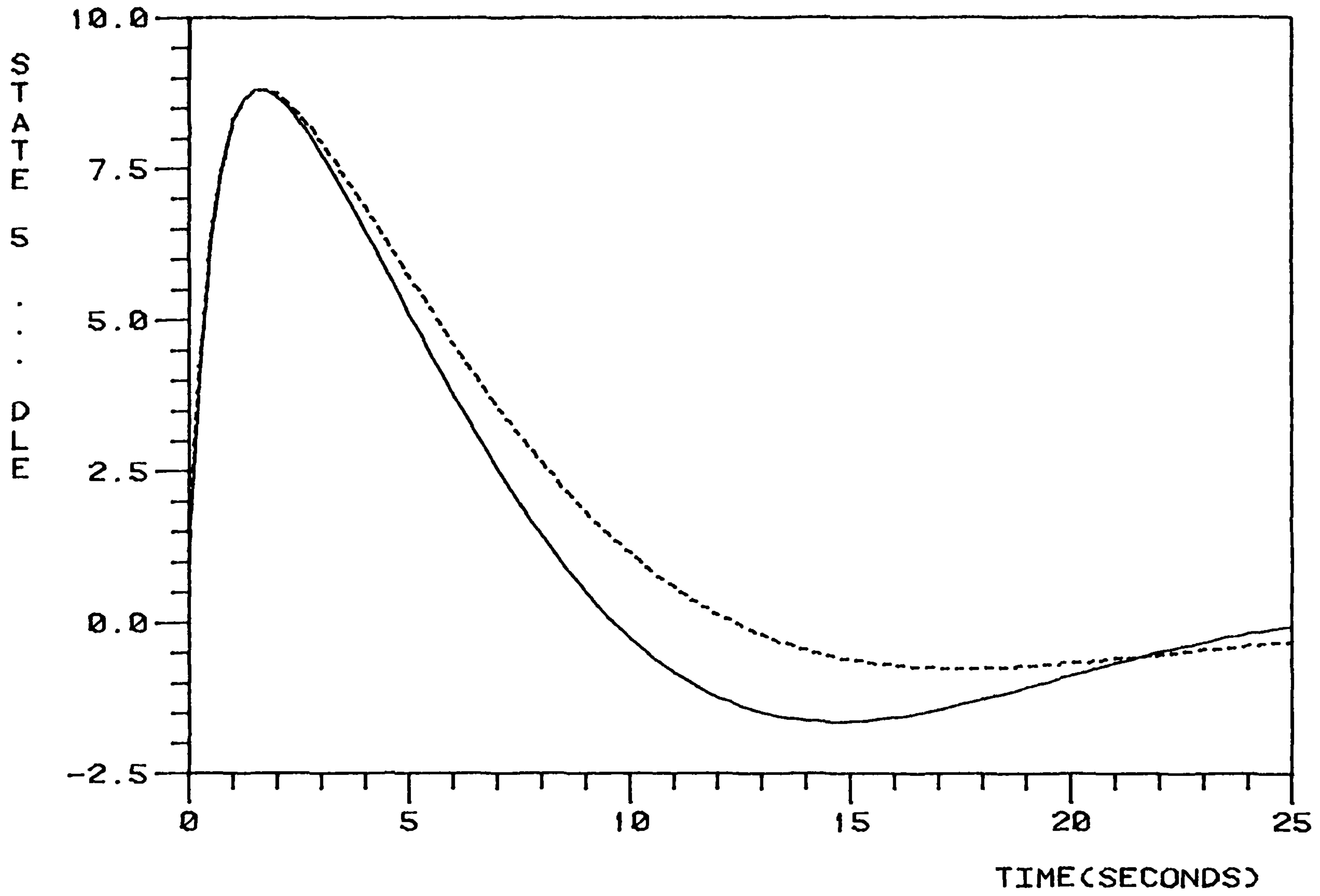

Figure 1.16. State 5 and added components using left eigenspace iterations.

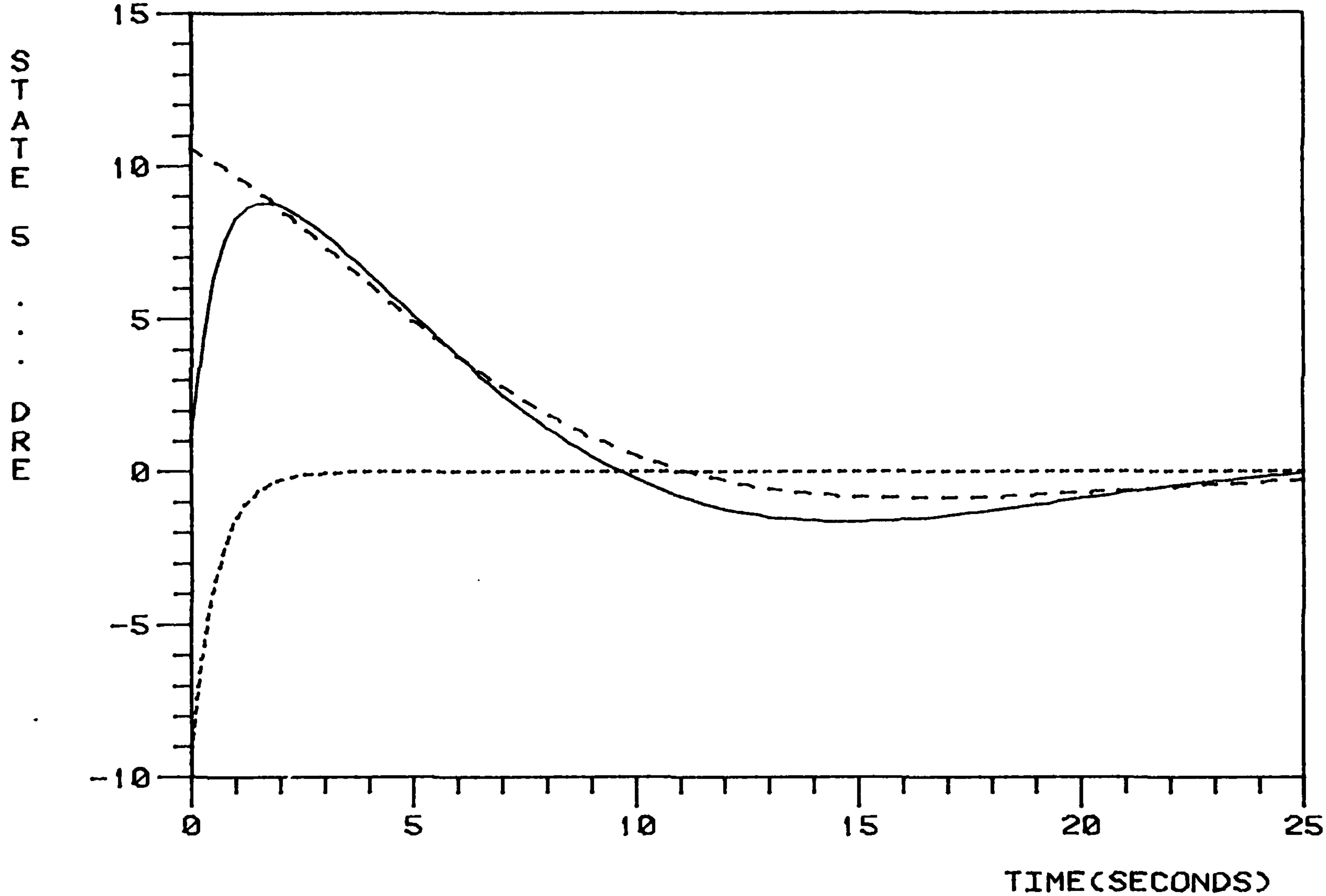

Figure 1.17. State 5 and components using right eigenspace iterations.

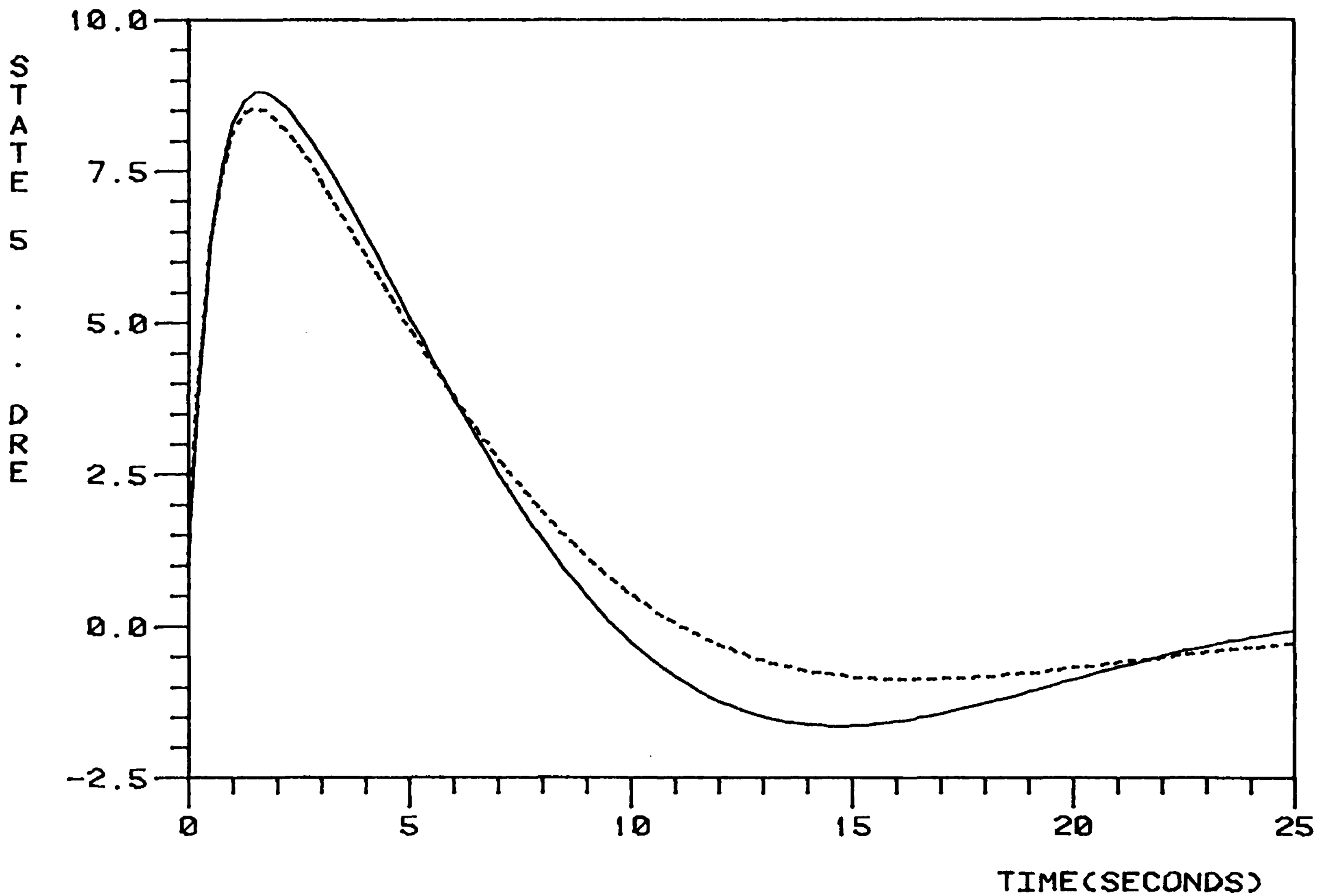

Figure 1.18. State 5 and added components using right eigenspace iterations.

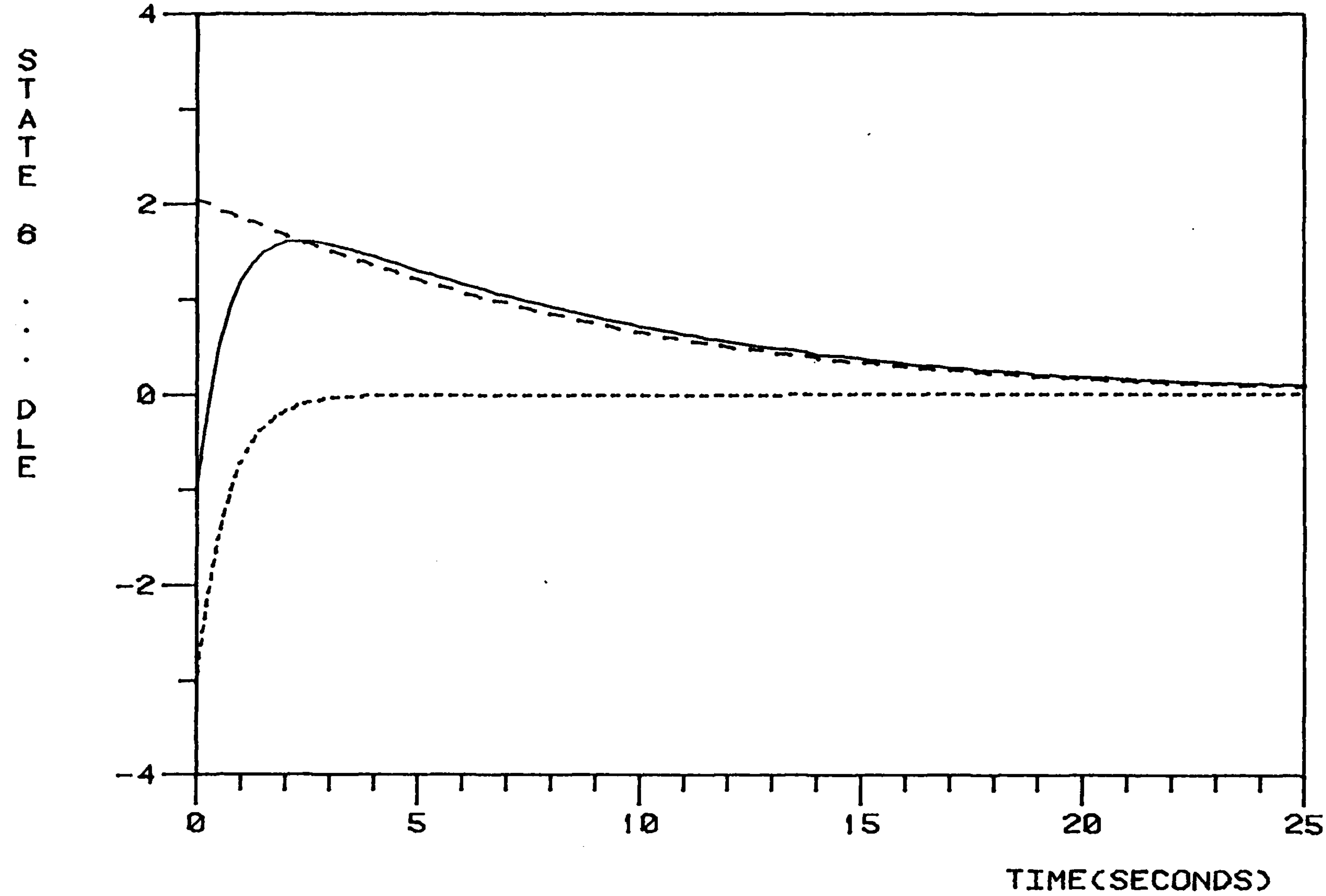

Figure 1.19. State 6 and components using left eigenspace iterations.

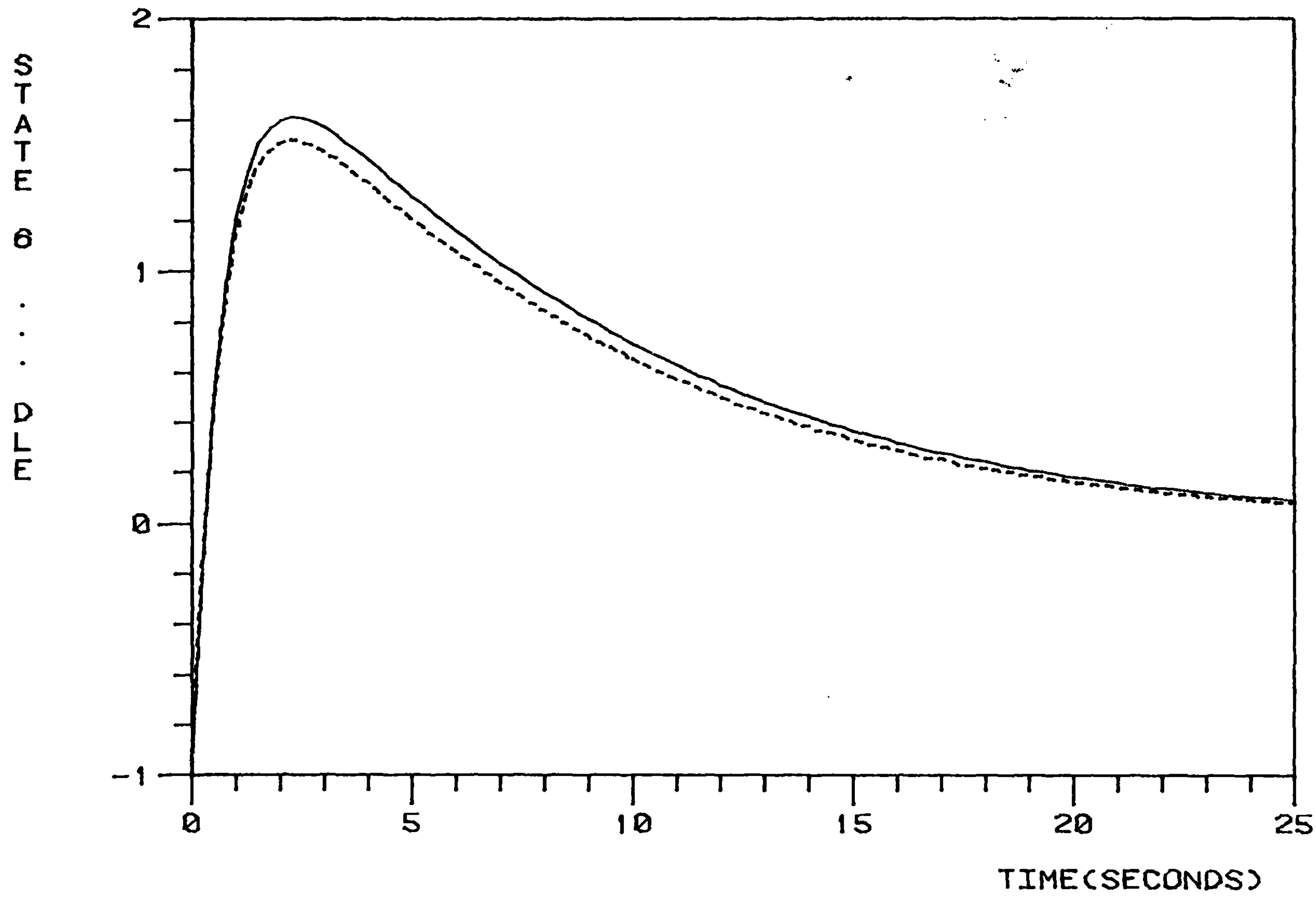

Figure 1.20. State 6 and added components using left eigenspace iterations.

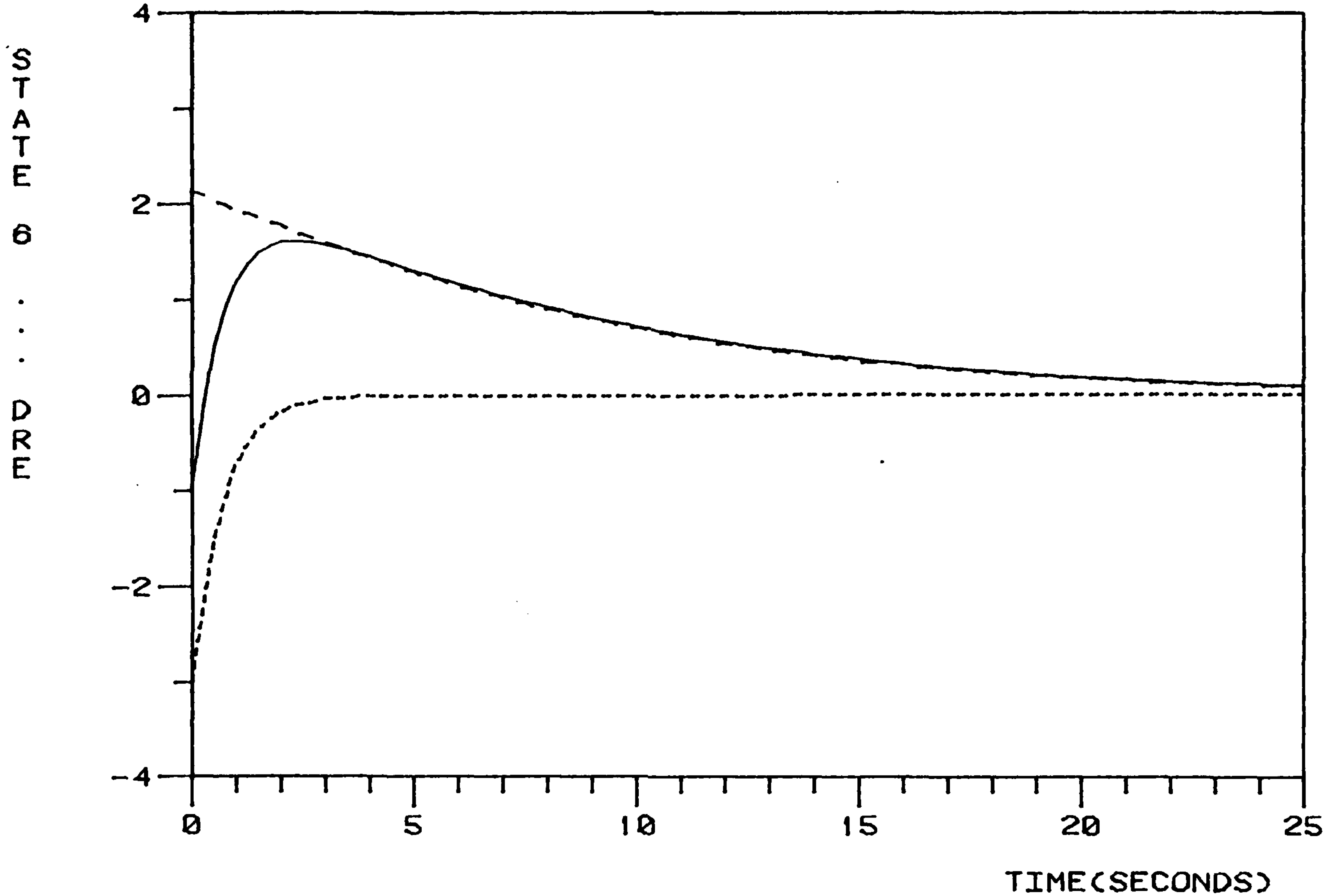

Figure 1.21. State 6 and components using right eigenspace iterations.

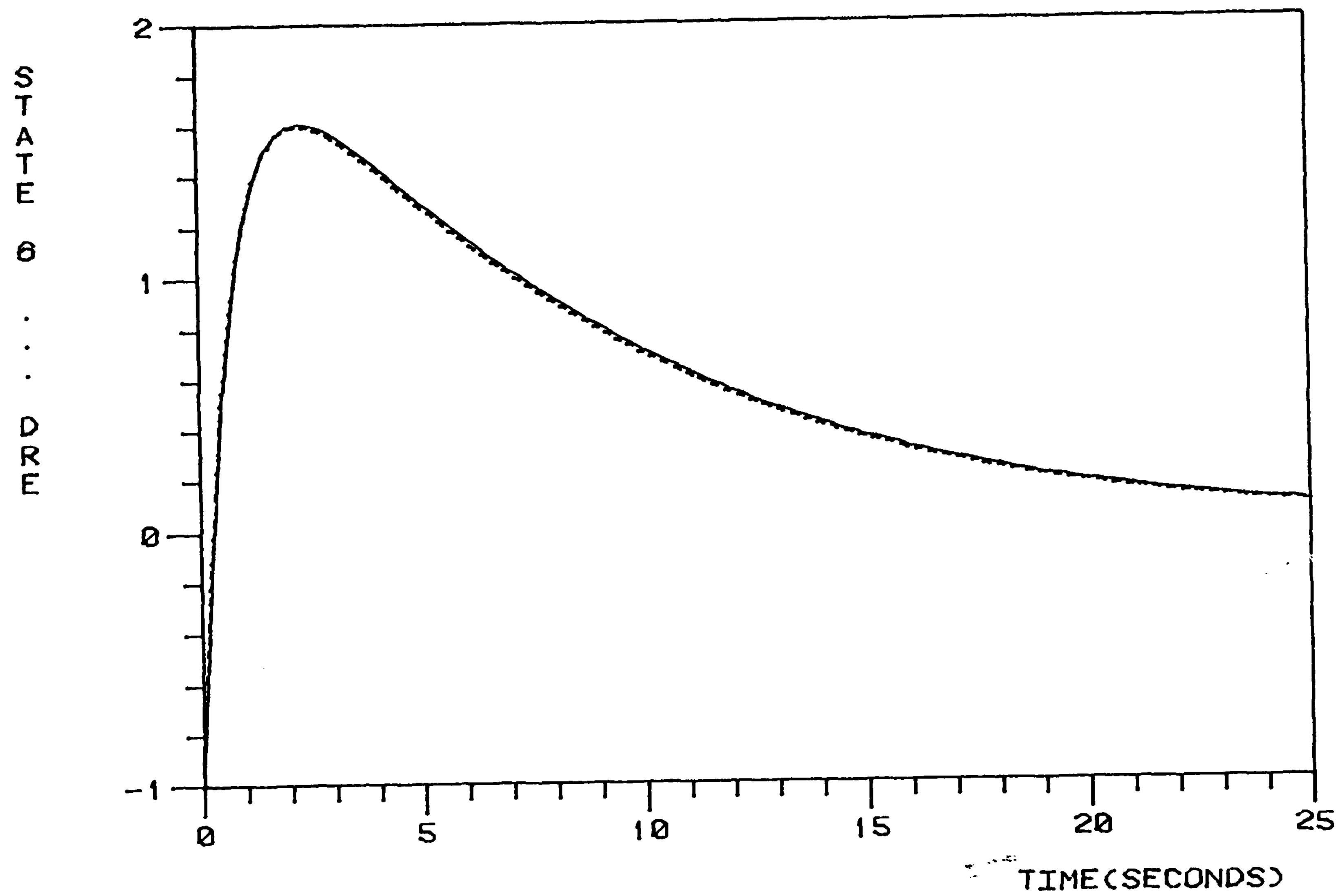

Figure 1.22. State 6 and added components using right eigenspace iterations.

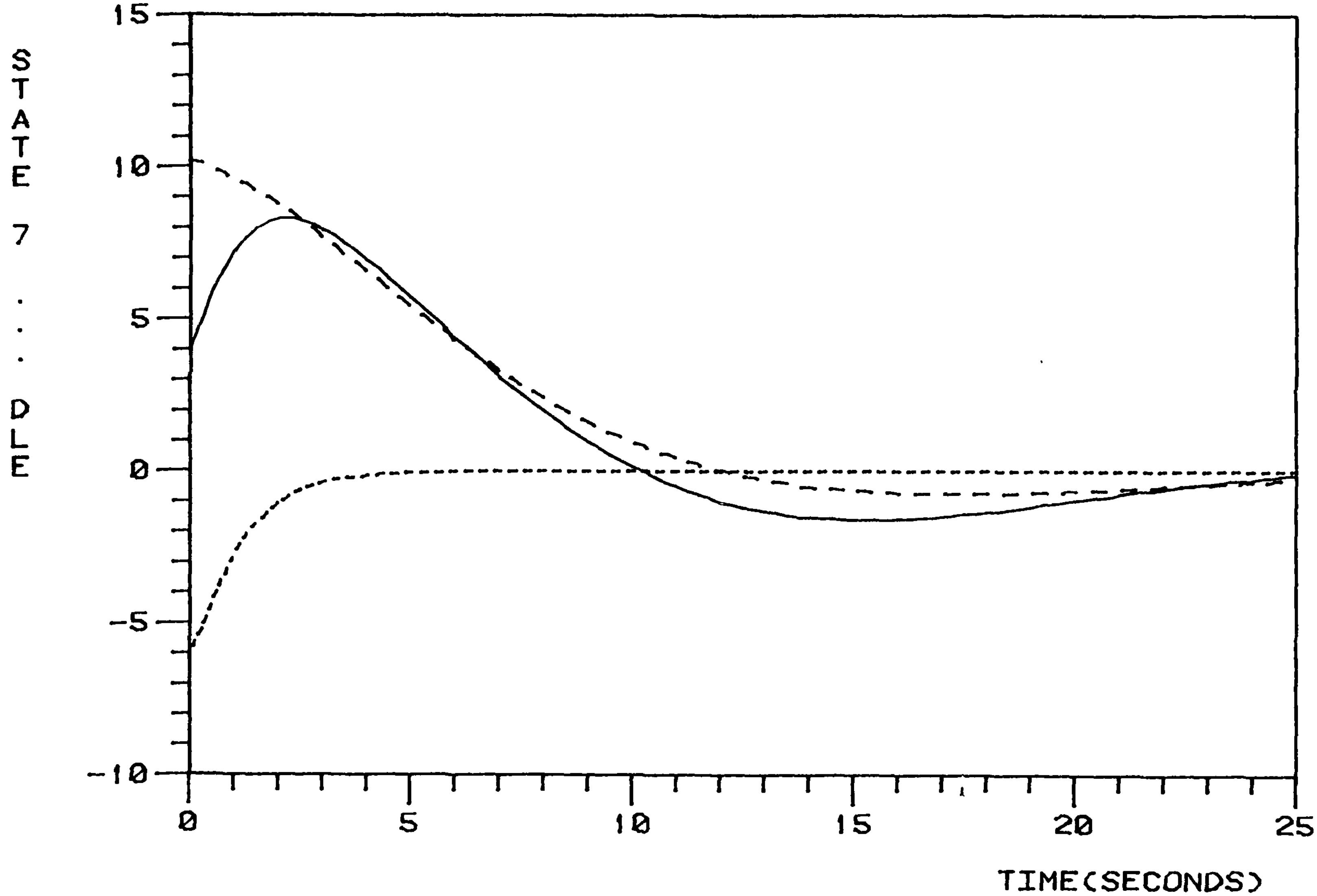

Figure 1.23. State 7 and components using left eigenspace iterations.

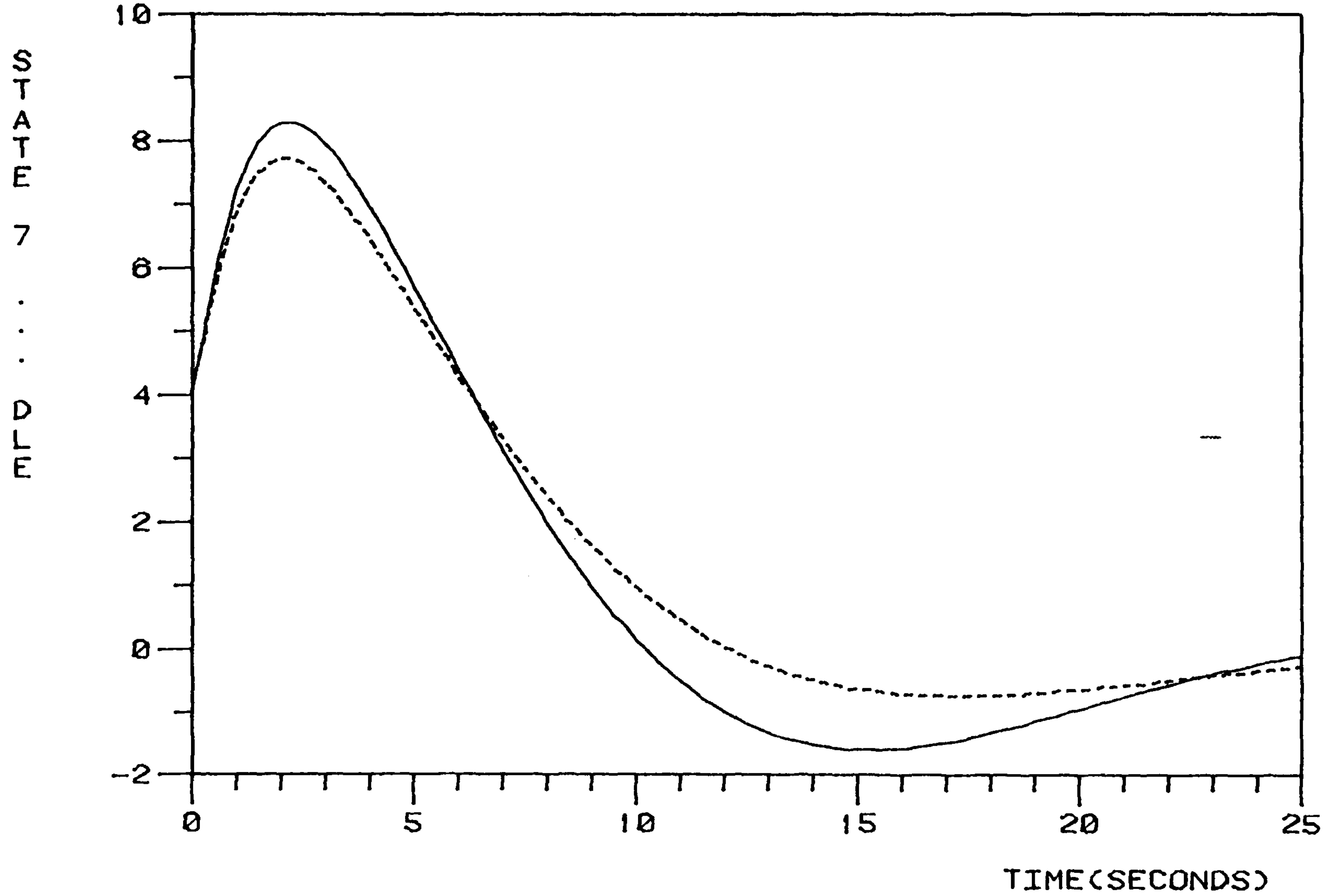

Figure 1.24. State 7 and added components using left eigenspace iterations.

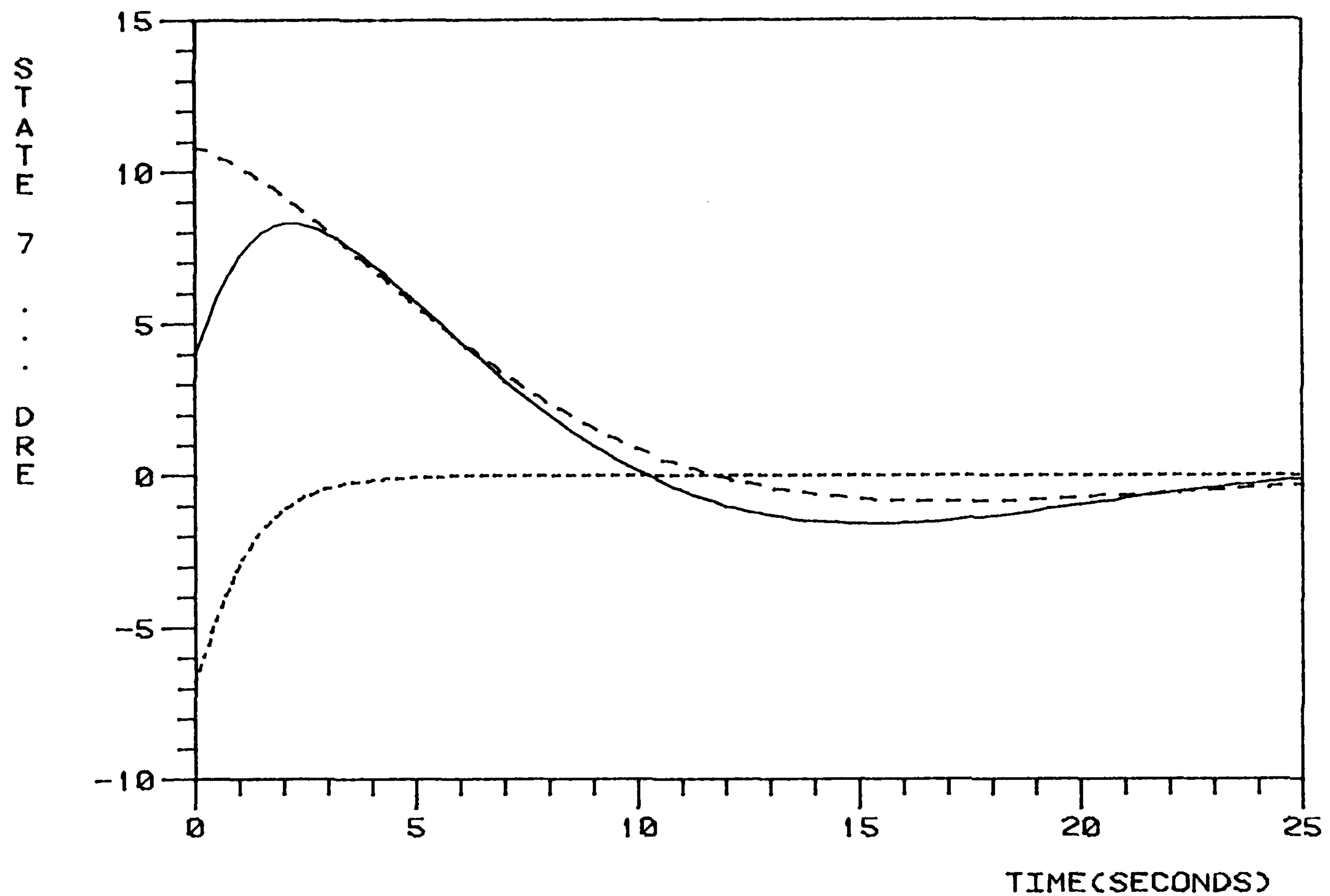

Figure 1.25. State 7 and components using right eigenspace iterations.

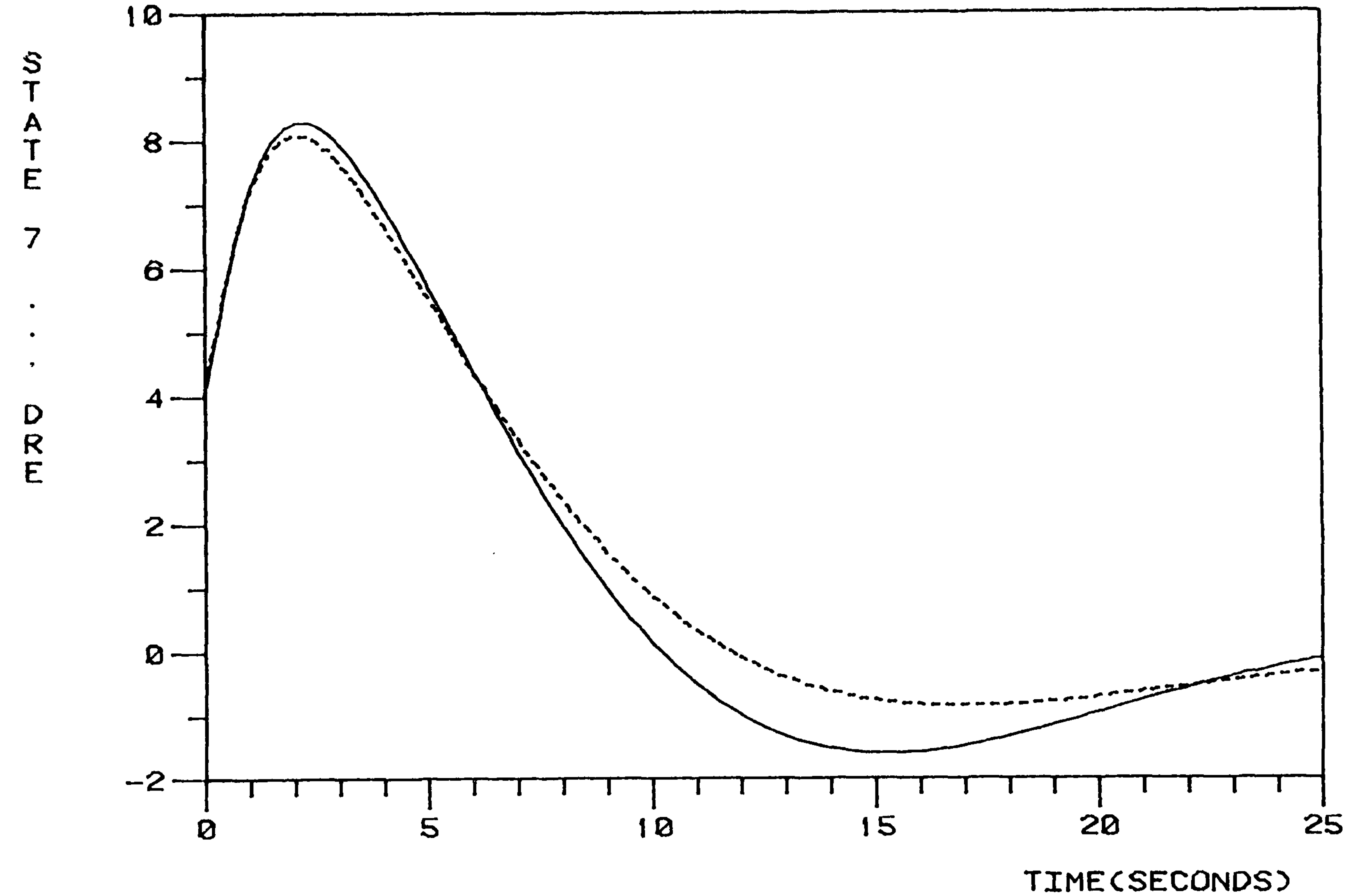

Figure 1.26. State 7 and added components using right eigenspace iterations.

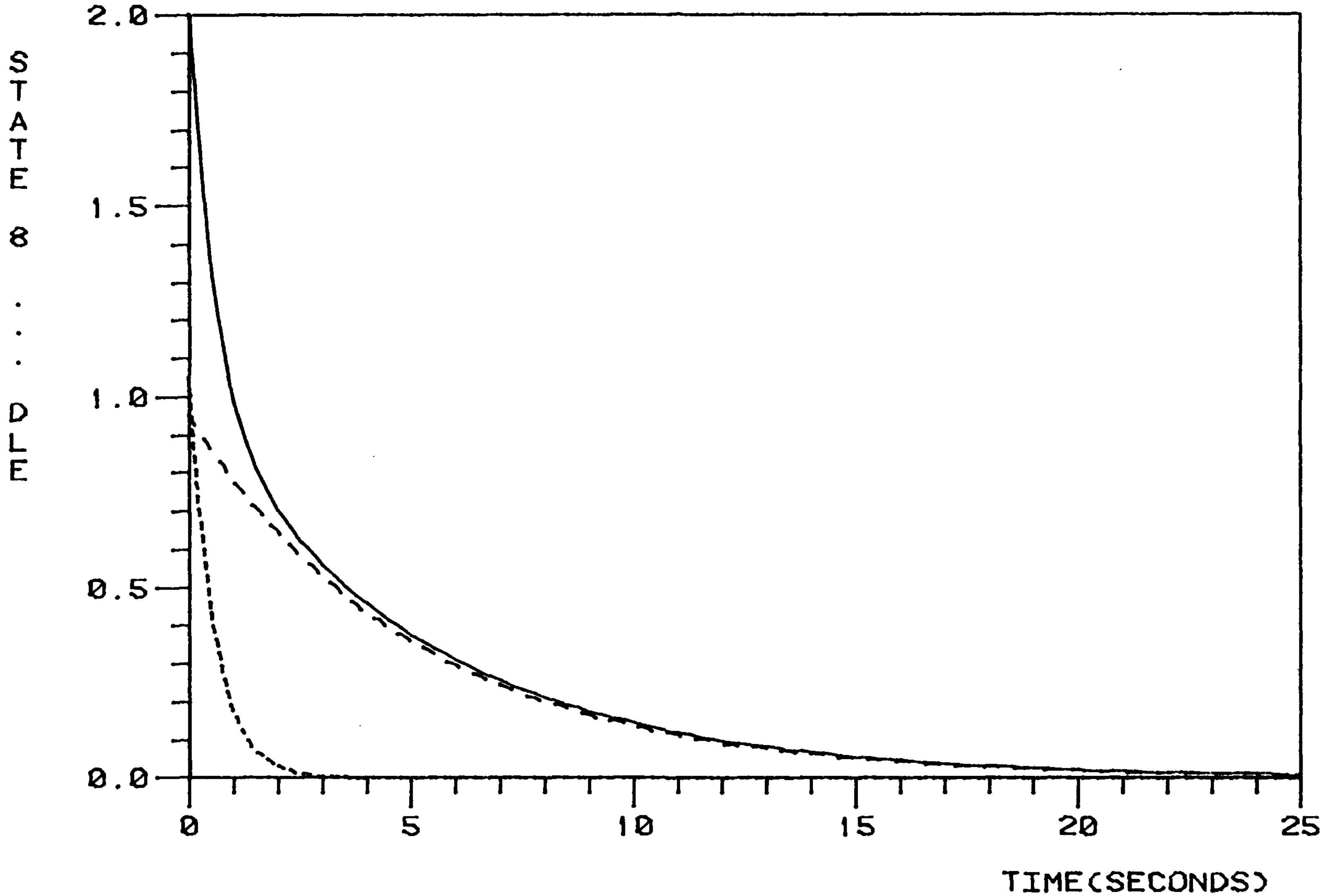

Figure 1.27. State 8 and components using left eigenspace iterations.

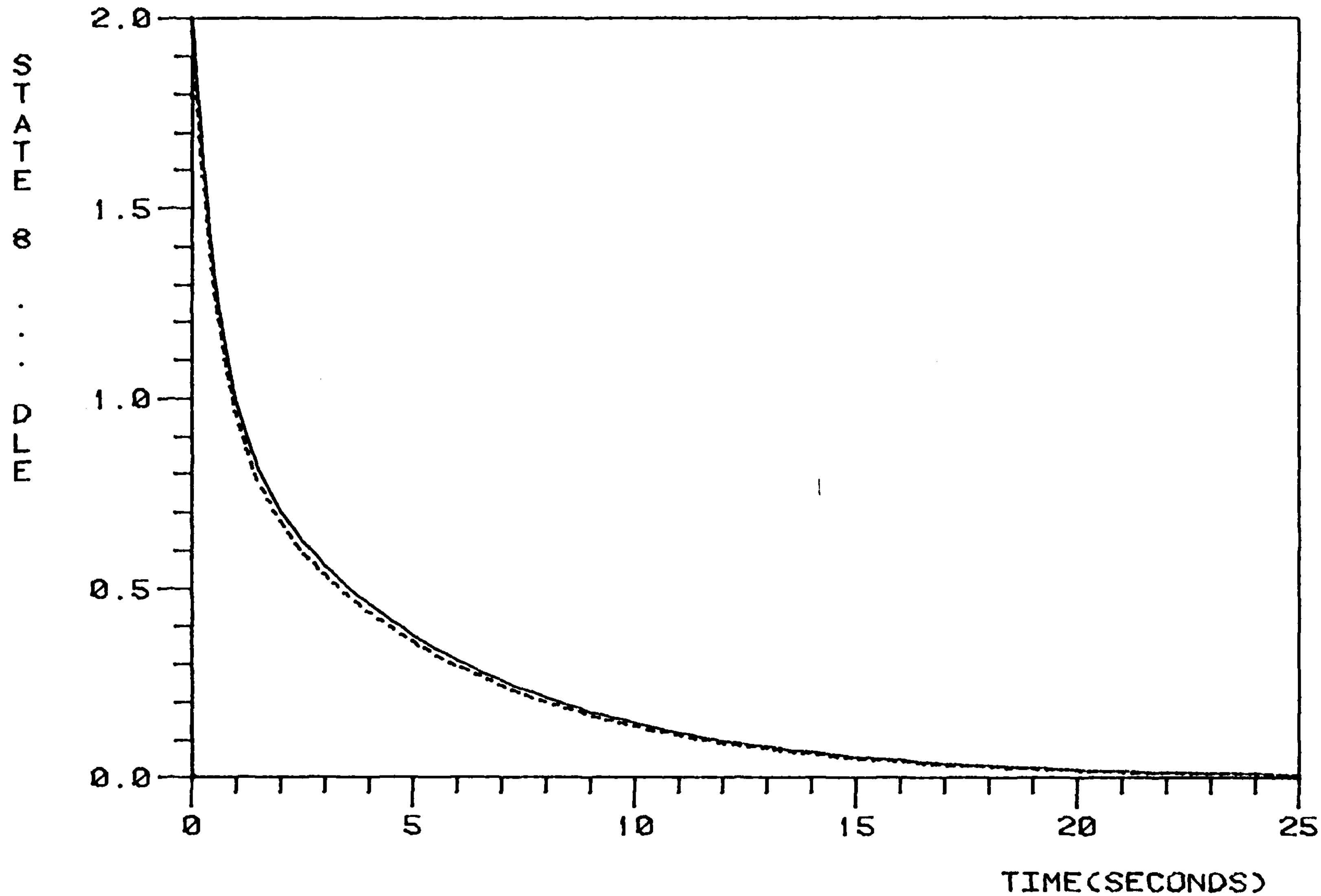

Figure 1.28. State 8 and added components using left eigenspace iterations.

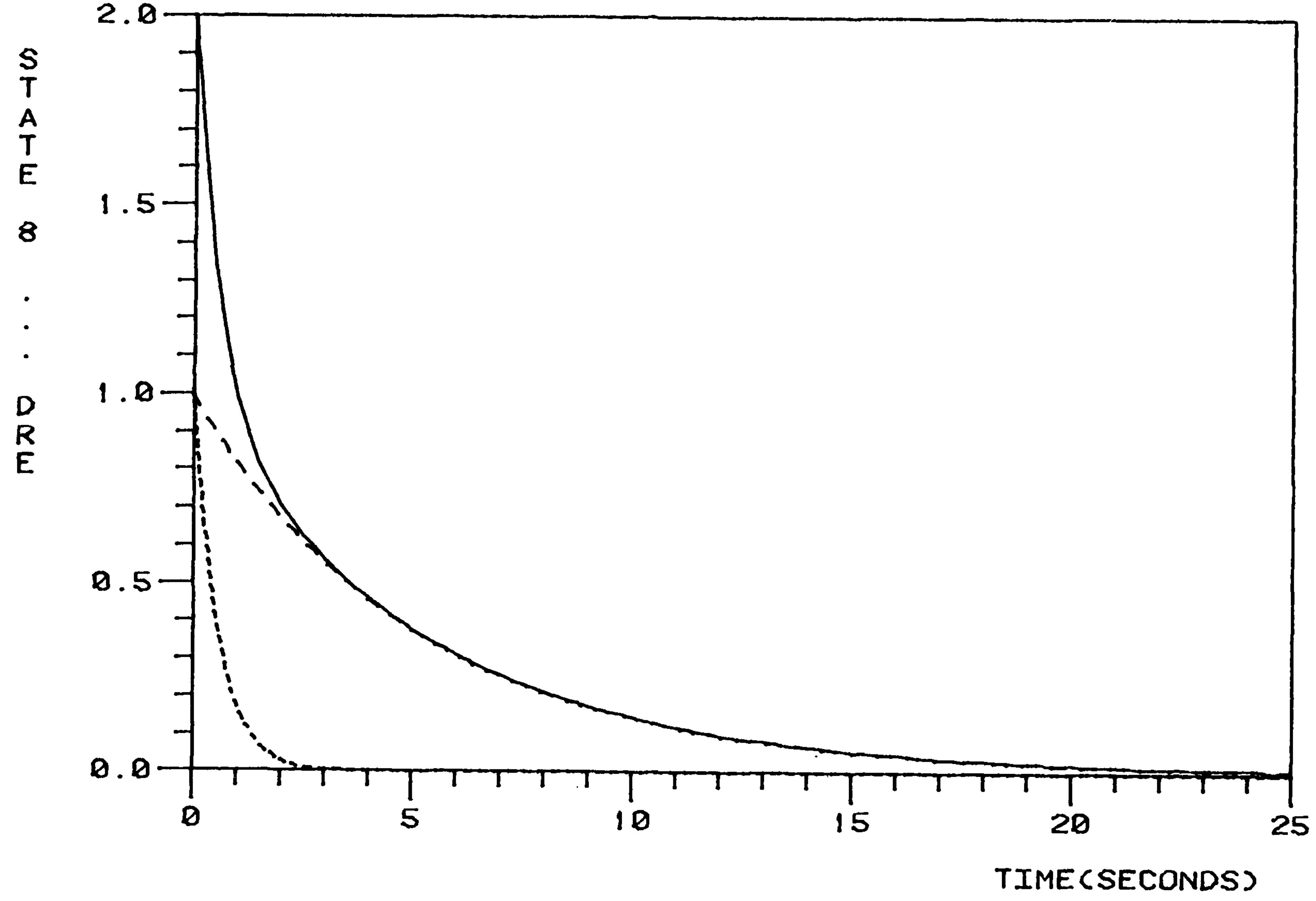

Figure 1.29. State 8 and components using right eigenspace iterations.

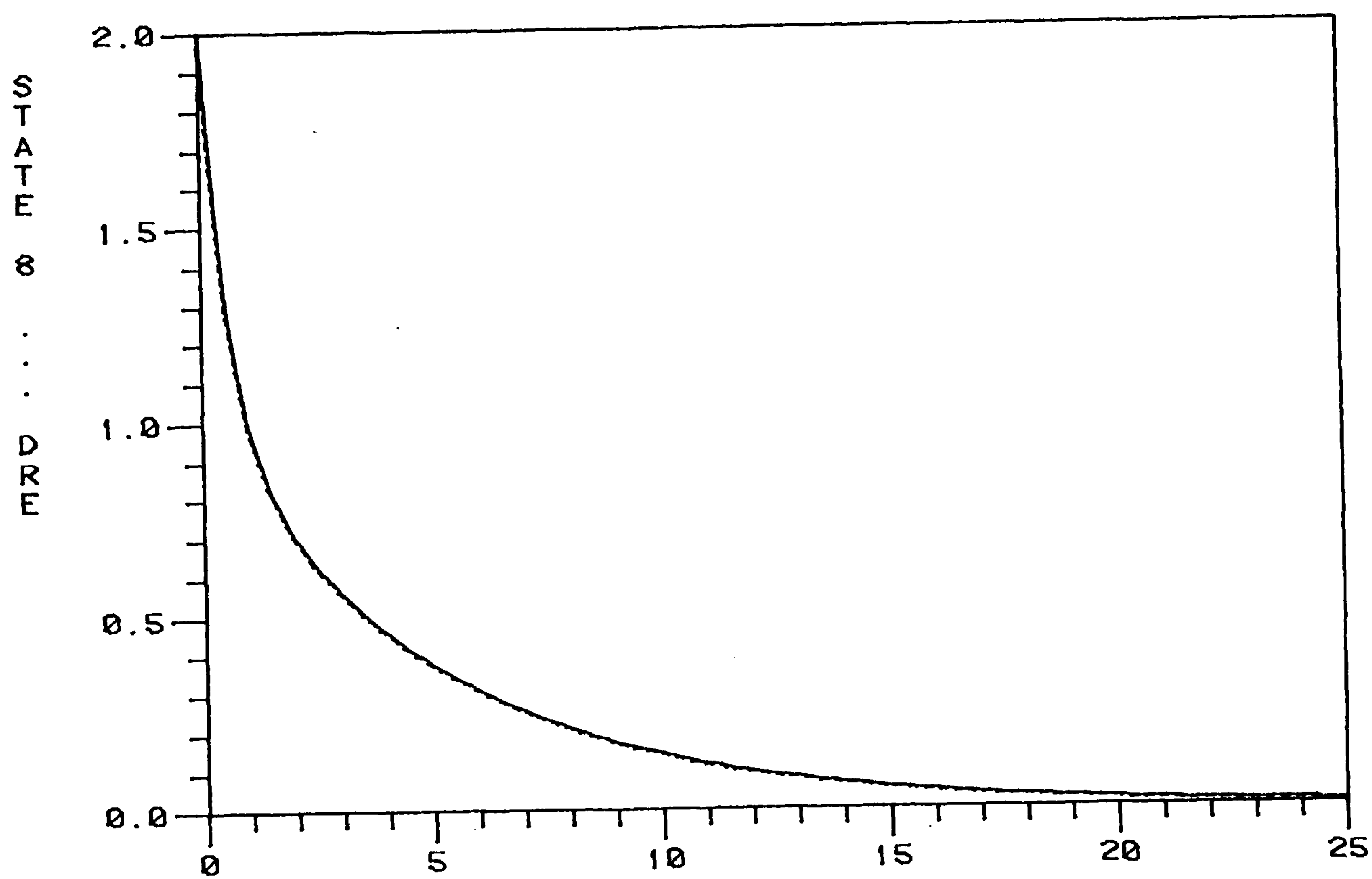

Figure 1.30. State 8 and added components using right eigenspace iterations.

II. ASYMPTOTIC SERIES DECOMPOSITION OF TIME-SCALES IN LINEAR TIME-INVARIANT SYSTEMS

A. Introduction

When small parameters are present in differential equations defining initial or boundary value problems, one of the popular methods of solution [15] is to obtain an asymptotic power series expansion of the solution. Such techniques have been well documented and can produce approximate solutions to problems where otherwise analytic explicit solution are impossible or exact numerical solutions are computationally not practical. Such systems are of the form

$$\dot{X} = f(X,t,\epsilon)X(0) = X_o \tag{2.1}$$

and a solution is sought of the form

$$X(t) = X^o(t) + \epsilon X^1(t) + \ldots.. \tag{2.2}$$

when such an expansion converges uniformly in X as $\epsilon \to 0$ we have a regular perturbation problem [16]. If there is a region of nonuniformity, usually at one of the boundaries, we have a singular perturbation problem. In most cases, the dynamics of the solution vector within this region of nonuniform convergence involve fast transients or the so called "Boundary Layer Phenomena". Thus, such singularly perturbed systems [16] are said to possess an inherent two-time-scale property characterized by a steady state or outer solution which is defined by the regions of uniform convergence of (2.2), and the boundary layer or inner solution where a stretched time variable is usually introduced in order to achieve convergence on the total time interval. In the linear case such systems take the form

$$\begin{aligned} \dot{y} &= Ay + Bz \quad && y(t) = y_o \\ \mu\dot{z} &= Cy + Dz \quad && z(t_o) = z_o \end{aligned} \tag{2.3}$$

Much work has been done to exploit the multiple time scale property of (1.1) when designing regulators, pole placement, reduced order modeling, etc. [17,18,19]. As a result, for a system which is known to have fast and slow phenomena, the systems engineer is motivated to permute the state in order to attain the above structure and take advantage of these decomposition techniques. It is the purpose of this chapter to use multiple time scale asymptotic expansions to obtain a "Steady State" and "Boundary Layer" decomposition in (2.3) and see how this decomposition compares to our eigenstructure decompositions of Chapter 1.

In Section B we obtain power series representations of our dichotomic transformation matrices P and $\hat{P}$. First an asymptotic power series is derived, then its equivalence to a convergent MacLauren series is established.

In Section C we derive important relationships between various fundamental sets of solution of (2.3) and any system satisfying (1.21). These fundamental sets are based on our reduced order subsystem matrices and the dichotomic transformation matrices P, $\hat{P}$, Q, and $\hat{Q}$.

In Section D we use Vasil'evas method of matched asymptotic expansions to obtain the "Boundary Layer" and "Steady State" components of the solution vectors y(t) and z(t). It is shown that this decomposition is equivalent to the eigenstructure decompositions of Chapter 1 by using one of the fundamental matrices established in Section C.

Section E discusses some computational simplifications to the dominant left and right eigenspace iterations based on system (2.3). The

simplifications involve eliminating the necessity to take an inverse at every iteration. The simolified iterations were originally proposed by [4] where a contraction mapping argument was used to obtain bounds for convergence. Using a different approach, less conservative bounds are obtained.

Finally, in Section F we give an example which highlights many of the important results of this Chapter.

B. Series Solutions to Ricatti Iterations

For proper spectral decomposition and dimensions m and n in (1), it was shown in section C of Chapter 1, that the following matrix recursion equation

$$P_{k+1} = P_k - (D + P_k B)^{-1} \cdot (DP_k - C + P_k B P_k - P_k A) \tag{2.4}$$

$$P_o = D^{-1} C$$

will converge to the dichotomic solution of

$$R(P) = DP - C + PBP - PA = 0 \tag{2.5}$$

when (1.1) is in the form of a singularly perturbed model, (2.4) becomes

$$P_{k+1} = P_k - (D + \mu P_k B)^{-1} \cdot (DP_k - C + \mu P_k B P_k - \mu P_k A) \tag{2.6}$$

$$P_o = D^{-1} C$$

In a later section in this chapter, we will use the method of matched asymptotic expansions in an attempt to decompose (2.3) into "steady-state" and "boundary layer" subsystems. It will be necessary in this derivation to obtain a power series solution to (2.5) in the form

$$P = P^o + \mu P^1 + \mu^2 P^2 + \ldots.. \tag{2.7}$$

From (2.6), it is important to note that such an expansion converges as $\mu \to 0$ uniformly in K. Thus, we have a regular perturbation problem [16].

Expressing (2.6) in a more convienient form

$$(D + \mu P_k B) P_{k+1} = C + \mu P_k A \tag{2.8}$$

We now substitute in the formal series (2.7) and obtain

$$[(D+\mu(P_k^o+\mu P_k^1+\ldots\ldots)B)(P_{k+1}^o+\mu P_{k+1}^1+\ldots\ldots)]$$

$$= C+\mu(P_k^o+\mu P_k^1+\ldots)A$$

Equating like powers of μ we obtain the so called "equations of the variations" [20].

$$DP_{k+1}^o = C \tag{2.9}$$

$$DP_{k+1}^1 + P_k^o B P_{k+1}^o = P_k^o A \tag{2.10}$$

$$DP_{k+1}^2 + P_k^o B P_{k+1}^1 + P_k^1 B P_{k+1}^o = P_k^1 A \tag{2.11}$$

$$\vdots$$

$$DP_{k+1}^N + \sum_{j=0}^{N-1} P_k^{N-1-j} B P_{k+1}^j = P_k^{N-1} A \tag{2.12}$$

Since D^{-1} exists, we can solve (2.9) as

$$P_{k+1}^o = D^{-1}C \tag{2.13}$$

and since our matching condition implies

$$P_o^o = D^{-1}C$$

$$P_o^k = 0,\ k > 0 \tag{2.14}$$

The solution to (2.13) is thus,

$$P_k^o = D^{-1}C \qquad \forall k \geq 0 \tag{2.15}$$

Likewise, the solution to (2.10) is found from

$$P^1_{k+1} = - D^{-1}P^o_k B P^o_{k+1} + D^{-1}P^o_k A \tag{2.16}$$

However, since $P^o_{k+1} = P^o_k \; \forall k \geq 0$

$$P^1_{k+1} = - D^{-1}P^o_k B P^o_k + D^{-1}P^o_k A = \text{constant matrix}$$

Thus,

$$P^1_k = 0 \qquad k = 0$$

$$= D^{-1}P^o_k B P^o_k + D^{-1}P^o_k A \qquad \forall k \geq = 1 \tag{2.17}$$

Continuing in this manner, the equilibrium solution for the N^{th} variation becomes

$$P^N_k = - D^{-1}\sum_{j=0}^{N-1} P^{N-1-j}_k B P^j_k + D^{-1}P^{N-1}_k A \tag{2.18}$$

Thus, we have a well defined method of generating all terms in the series. However, to prove that the series (2.7) does asymptotically solve the matrix recursion (2.8) as $\mu \to 0$, let

$$P(\mu) = \sum_{j=0}^{N} P^j \mu^j + R(\mu) \qquad N \geq 0 \tag{2.19}$$

We shall show that the remainder term is unique and $O(\mu^{N+1})$.

Substituting (2.19) into (2.8)

$$D(P^o_{k+1} + \mu P^1_{k+1} + \ldots\ldots + \mu^N P^N_{k+1} + R_{k+1}(\mu))$$

$$+ \mu(P^o_k + \mu P^1_k + \ldots + \mu^N P^N_k + R_k(\mu))B(P^o_{k+1} + \mu P^1_{k+1} + \ldots + \mu^N P^N_{k+1} + R_{k+1}(\mu)$$

$$= \mu(P^o_k + \mu P^1_k + \mu^2 P^2_k + \ldots + \mu^N P^N_k + R_k(\mu))A + C \tag{2.20}$$

Using the series solutions to P^j, j = 1, N, (2.20) reduces to

$$DR_{k+1}(\mu) + \mu R_k(\mu) BP_{k+1}(\mu) + \mu(P_k^o BR_{k+1}(\mu) + P_k(\mu)BP_{k+1}^o)$$

$$+ \mu^2(P_k^1 BR_{k+1}(\mu) + R_k(\mu)BP_{k+1}^1)$$

$$+$$

$$\vdots$$

$$+ \mu^{N+1}(P_k^N BR_{k+1}(\mu) + R_k(\mu)BP_{k+1}^N)$$

$$+ \mu^{N+1} f(P_k^o, P_k^1, \;\ldots\; P_k^N) = \mu R_k(\mu)A \tag{2.21}$$

Collecting terms,

$$R_{k+1}(\mu) = (D + \mu(P_k^o + \mu P_k^1 + \ldots + \mu^N P_k^N)B + \mu R_k)^{-1}.$$

$$[\mu R_k(\mu)B(P_{k+1}^o + \mu P_{k+1}^1 + \ldots + \mu^N P_{k+1}^N) + \mu R_k(\mu)A]$$

$$+ O(\mu^{N+1}) \tag{2.22}$$

However, since $R_o = 0$, (2.22) can be uniquely solved recursively

i.e. $$R_1 = (D + \mu P_o^o B)^{-1} \cdot 0 + O(\mu^{N+1}) = O(\mu^{N+1})$$

$$R_2 = (D + \mu \sum_{j=0}^{N} P_1^j B + O(\mu^{N+2}))^{-1} O(\mu^{N+2})$$

$$+ O(\mu^{N+1}) = O(\mu^{N+1})$$

and the unique solution is given by

$$R(\mu) = \lim_{k\to\infty} R_k(\mu) = O(\mu^{N+1}) \tag{2.23}$$

Thus, the solution to (2.22) can be expressed as

$$P(\mu) = \sum_{j=0}^{N} \mu^j P^j + O(\mu^N) \tag{2.24}$$

$\forall N \geq 0$

or using the notation of [15],

$$P(\mu) \sim \sum_{j=0}^{\infty} \mu^j P^j \tag{2.25}$$

Asymptotic power serie like (2.25), in general do not converge for any fixed nonzero value of μ. If however, we can show the function $P_k(\mu)$ to be holomorphic in μ in some neighborhood of the origin in the complex μ plane, then we know that for sufficiently small μ, $P(\mu)$ possesses a convergent MacLauren series expansion. And by the uniqueness of this series, it is equivalent to (2.25) [15].

<u>Lemma 5</u>

Let $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in C^{N\times N}$ and $\mu \in C$.

If F satisfies property (1.21), then there exists a neighborhood S of the orgin in μ space such that $P_k(\mu)$ is holomorphic in S $\forall k \geq 0$.

Proof:

To prove analyticity of P_k in a define region S of the μ plane, it is both necessary and sufficient to show

$$\frac{\partial P_k}{\partial \mu} \text{ exists, and is continuous in S}$$

We show this by induction

$P_o = D^{-1}C =$ constant function, entire

$P_1 = (D + \mu P_o B)^{-1} \cdot (\mu P_o A + C)$

$$\frac{\partial P_1}{\partial \mu} = (D + \mu P_o B)^{-1} \cdot (P_o A + \mu \frac{\partial P_o}{\partial \mu} A)$$

$$+ (D + \mu P_o B)^{-1} (P_o B + \mu \frac{\partial P_o}{\partial \mu} B)(D + \mu P_o B)^{-1} \cdot (\mu P_o A + C)$$

The analyticity of $P_o \forall \mu$ implies analyticity of P_1 if

$$0 \le \mu < \frac{1}{\|D^{-1} P_o B\|} \tag{2.26}$$

Now assume $\frac{\partial P_k}{\partial \mu}$ exists and is continuous for

$$0 \le \mu \le \frac{1}{\|D^{-1} P_{k-1} B\|} .$$

Then,

$$\frac{\partial P_{k+1}}{\partial \mu} = (D + \mu P_k B)^{-1} \cdot (P_k A + \mu \frac{\partial P_k}{\partial \mu} A)$$

$$+ (D + \mu P_k B)^{-1} \cdot (P_k B + \mu \frac{\partial P_k}{\partial \mu} B)(D + \mu P_k B)^{-1} (\mu P_k A + C)$$

which exists and is continuous if

$$0 \le \mu < \frac{1}{\|D^{-1}P_k B\|}$$

From Chapter 1, if condition (1.21) is satisfied, then P_k converges to the dichotomic solution. Thus, there exists a constant C such that

$$\|P_k\| \le C \qquad \forall k \ge 0$$

Thus, for

$$0 \le \mu < \frac{1}{\|D^{-1}\| \, \|B\| C} \tag{2.27}$$

P_k is a holomorphic function of μ $\forall k \ge 0$

Thus, P_k possesses a unique convergent MacLauren series given by (2.25) for every μ bounded by (2.27) $\forall k \ge 0$.

In a completely analogus manner, the series solution to

$$\hat{P}_{k+1} = \hat{P}_k + (\mu B + \mu A\hat{P}_k - \hat{P}_k D - \hat{P}_k C \hat{P}_k)\cdot(D + C\hat{P}_k)^{-1} \tag{2.28}$$

$$\hat{P}_o = \mu B D^{-1}$$

can be obtained in the form

$$\hat{P} = \hat{P}^o + \mu\hat{P}^1 + \ldots.. \tag{2.29}$$

where

$$\hat{P}^o = 0 \tag{2.30}$$

$$\hat{P}^1 = BD^{-1} \tag{2.31}$$

$$\vdots$$

$$\hat{P}^N = (-\sum_{j=1}^{N-1} P^{N-j} C P^j)\cdot D^{-1} + AP^{N-1}D^{-1} \tag{2.32}$$

and likewise, (2.29) is convergent for

$$0 \le \mu < \frac{1}{\|D^{-1}\| \; \|C\|\hat{C}} \tag{2.33}$$

where $\hat{C} \le \frac{\|\hat{P}_k\|}{\mu} \; \forall k \ge 0$, since $\|\hat{P}_k\| = 0(\mu)$.

C. Fundamental Sets of Solutions

In this section we develop some basic properties relating the dichotomic dominant left and right eigenspace transformations of Chapter 1. The need for these properties will become apparent in the next section.

One of the basic properties of linear homogeneous systems of differential equations of the form

$$\dot{x} = Fx \tag{2.34}$$

is that a fundamental set of solutions [21] is of the form

$$X(t) = e^{Ft} \tag{2.35}$$

For a given initial value problem $x(t_o) = x_o$, the solution to (2.34) for $t \geq t_o$ is uniquely given by

$$x(t) = X(t)X(t_o)^{-1}x_o \tag{2.36}$$

or when using (2.35)

$$x(t) = e^{F(t-t_o)}x_o$$

Another property of the fundamental matrix (2.35) is that given any nonsingular matrix M,

$$Y(t) = e^{Ft} \cdot M \tag{2.37}$$

is also a fundamental matrix.

For system (1.1) satisfying condition (1.21) we have established transformation matrices P, Q, $\hat{P}$, and $\hat{Q}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} \begin{bmatrix} A-BP & 0 \\ 0 & D+PB \end{bmatrix} \begin{bmatrix} I-QP & -Q \\ P & I \end{bmatrix} \tag{2.38}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I-\hat{P}\hat{Q} & \hat{P} \\ -\hat{Q} & I \end{bmatrix} \begin{bmatrix} A-\hat{P}C & 0 \\ 0 & D+C\hat{P} \end{bmatrix} \begin{bmatrix} I & -\hat{P} \\ \hat{Q} & I-\hat{Q}\hat{P} \end{bmatrix} \tag{2.39}$$

Thus,

$$e^{\begin{bmatrix} A & B \\ C & D \end{bmatrix} t} = \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} \begin{bmatrix} e^{(A-BD)t} & 0 \\ 0 & e^{(D+PB)t} \end{bmatrix} \begin{bmatrix} I-QP & -Q \\ P & I \end{bmatrix} \tag{2.40}$$

is a fundamental matrix for (1.1).

Thus, by (2.37), so is

$$e^{\begin{bmatrix} A & B \\ C & D \end{bmatrix} t} \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} = \begin{bmatrix} I & Q \\ -P & I-PQ \end{bmatrix} \begin{bmatrix} e^{(A-BP)t} & 0 \\ 0 & e^{(D+PB)t} \end{bmatrix} \tag{2.41}$$

or, the columns of

$$X(t) = \begin{bmatrix} e^{(A-BP)t} & Qe^{(D+PB)t} \\ -Pe^{(A-BP)t} & (I-PQ)e^{(D+PB)t} \end{bmatrix} \tag{2.42}$$

form a fundamental set for the system (1.1). Likewise, using a similar argument for (2.39), the columns of

$$X(t) = \begin{bmatrix} (I-\hat{P}\hat{Q})e^{(A-\hat{P}C)t} & \hat{P}e^{(D+C\hat{P})t} \\ -\hat{Q}e^{(A-\hat{P}C)t} & e^{(D+C\hat{P})t} \end{bmatrix} \tag{2.43}$$

Now, by the dichotomic property of the transformations, there exist nonsingular matrices T_1, T_2, T_3 and T_4 such that

$$T_1^{-1}(A-BP)T_1 = \Lambda_1 \tag{2.44}$$

$$T_2^{-1}(D+PB)T_2 = \Lambda_2 \tag{2.45}$$

$$T_3^{-1}(A-\hat{P}C)T_3 = \Lambda_1 \tag{2.46}$$

$$T_4^{-1}(D+C\hat{P})T_4 = \Lambda_2 \tag{2.47}$$

where Λ_2 is the dominant eigenvalue matrix and Λ_1 is the eigenvalue matrix consisting of the rest of the spectrum of (1.1).

Using (2.44), (2.45), (2.46), and (2.47) in (2.38) and (2.39) gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} (I\ \hat{P}\hat{Q})T_1 & \hat{P}T_2 \\ -\hat{Q}T_1 & T_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} T_1^{-1} & -T_1^{-1}\hat{P} \\ T_2^{-1}\hat{Q} & T_2^{-1}(I-\hat{Q}\hat{P}) \end{bmatrix} \tag{2.48}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} T_3 & QT_4 \\ -PT_3 & (I-PQ)T_4 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} T_3^{-1}(I-QP) & -T_3^{-1}Q \\ T_4^{-1}P & T_4^{-1} \end{bmatrix} \tag{2.49}$$

which, by definition identifies

$$\begin{bmatrix} (I-\hat{P}\hat{Q})T_1 & \hat{P}T_2 \\ -\hat{Q}T_1 & T_2 \end{bmatrix}, \quad \begin{bmatrix} T_3 & QT_4 \\ -PT_3 & (I-PQ)T_4 \end{bmatrix} \tag{2.50}$$

as eigenvector matrices for (1.1).

While the magnitudes of eigenvectors are not unique, there directions are. Thus,

$$\begin{bmatrix} \hat{P}T_2 \\ T_2 \end{bmatrix} e^{\Lambda_2 t}, \qquad \begin{bmatrix} T_3 \\ -PT_3 \end{bmatrix} e^{\Lambda_1 t} \tag{2.51}$$

also serve as a fundamental set for (1.1), or in matrix form

$$\begin{bmatrix} T_3 e^{\Lambda_1 t} & \hat{P}T_2 e^{\Lambda_2 t} \\ -PT_3 e^{\Lambda_1 t} & T_2 e^{\Lambda_2 t} \end{bmatrix} \tag{2.52}$$

However, postmultiplying (2.52) by the nonsingular matrix

$$\begin{bmatrix} T_3^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix} \tag{2.53}$$

gives (2.52) as

$$Y(t) = \begin{bmatrix} e^{(A-BP)t} & \hat{P}e^{(D+C\hat{P})t} \\ -Pe^{(A-BP)t} & e^{(D+C\hat{P})t} \end{bmatrix} \tag{2.54}$$

Also, by a similar argument on (2.48) and (2.49),

$$\hat{Y}(t) = \begin{bmatrix} (I-\hat{P}\hat{Q})e^{(A-\hat{P}C)t} & Qe^{(D+PB)t} \\ -\hat{Q}e^{(A-\hat{P}C)t} & (I-PQ)e^{(D+PB)t} \end{bmatrix} \tag{2.55}$$

also qualifies as a fundamantal matrix for (1.1).

Thus, in this section we have established the existence of four fundamental matrices for (1.1) based on the dichotomic transformation matrices P, Q, $\hat{P}$ and $\hat{Q}$. This flexibility will prove valuable in the next section concerning asymptotic expansions of our singularly perturbed model (2.3).

D. Solution by Asymptotic Expansion Via the Method of Vasil'eva

Using the results established in the first two sections of this chapter, we will attempt to solve (2.3) using asymptotic expansion techniques.

In [20], the method of matched asymptotic expansions was proposed as a method of obtaining asumptotic solutions to the general nonlinear singularly perturbed initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f(z,y,t) \qquad y(0) = y_o \\ \mu \frac{dz}{dt} &= F(z,y,t) \qquad z(0) = z_o \end{aligned} \tag{2.56}$$

To use this method, it is assumed that the root $z = \varphi(y,t)$ of the equation

$$F(y,z,t) = 0 \tag{2.57}$$

is stable in the first approximation or specifically, the real parts of the roots of the characteristic equation

$$\mathrm{DET}\left(\left. \frac{\partial F}{\partial z} \right|_{z=\varphi(y,t)} - \lambda I \right) = 0 \tag{2.58}$$

be negative in D, where D is a closed bounded domain in the variables $t_o \le t \le t_1$, $\|z\| < K_1$, $\|y\| < K_2$, and $0 \le \mu < \mu_o$. Under this assumption, the method can be applied to (2.3) as clearly carried out in [20]. In our case (2.56) reduces to

$$\begin{aligned} \frac{dy}{dt} &= Ag + Bz \qquad y(0) = y_o \\ \mu \frac{dz}{dt} &= Cy + Dz \qquad z(0) = z_o \end{aligned} \tag{2.59}$$

and the assumption becomes

$$Re(\lambda_i) < 0 \quad \forall \lambda_i \epsilon \sigma(D) \quad i=1, \ldots m$$

with this assumption satisfied, we can proceed with the asumptotic solution method proposed by Vasil'eva.

A solution to (2.59) is sought in the form

$$y = \bar{y} + \pi_y \tag{2.60}$$

$$z = \bar{z} + \pi_z \tag{2.61}$$

where

$$y = y_o(t) + \mu y_1(t) + \ldots\ldots\ldots\ldots \tag{2.62}$$

denotes a formal power series in μ whose coefficients depend on t, and

$$\pi_y = \pi_o y(\tau) + \mu\pi_1 y(\tau) + \ldots\ldots\ldots\ldots \tag{2.63}$$

Denotes a formal power series in μ whose coefficients depend on $\mu\tau = t/\mu$.

Substitution of (2.62), (2.63) and the analog bus expansions into (2.59) yields

$$\mu \frac{d\bar{y}}{dt} + \frac{d}{d\tau}\pi y = \mu A(\bar{y}+\pi y) + \mu B(z+\pi \ )$$

$$\tag{2.64}$$

$$\mu \frac{d\bar{z}}{dt} + \frac{d}{d\tau}\pi z = C(y+\pi y) + D(\bar{z}+\pi z)$$

Equating the coefficients of equal powers of μ, those depending on t and those depending on τ being treated separately, we arrive at the following equations for the variations.

Zeroth order,

$$C\bar{y}_o(t) + D\bar{z}_o(t) = 0 \tag{2.65}$$

$$\frac{d}{d\tau}\,\pi_o z = C\pi_o y + D\pi_o z \tag{2.66}$$

$$\frac{d\bar{y}_o(t)}{dt} = A\bar{y}_o(t) + B\bar{z}_o(t) \tag{2.67}$$

$$\frac{d}{d\tau}\,\pi_o y(\tau) = 0 \tag{2.68}$$

First order,

$$\frac{d\bar{z}_o(t)}{dt} = C\bar{y}_1(t) + D\bar{z}_1(t) \tag{2.69}$$

$$\frac{d\pi_1 z(\tau)}{d\tau} = C\pi_1 y(\tau) + D\pi_1 z(\tau) \tag{2.70}$$

$$\frac{d\bar{y}_1(t)}{dt} = A\bar{y}_1(t) + B\bar{z}_1(t) \tag{2.71}$$

$$\frac{d\pi_1 y(\tau)}{d\tau} = A\pi_o y(\tau) + B\pi_o z(\tau) \tag{2.72}$$

k^{th} order

$$\frac{d\bar{z}_{k-1}(t)}{dt} = C\bar{y}_k(t) + D\bar{z}_k(t) \tag{2.73}$$

$$\frac{d\pi_k z(\tau)}{d\tau} = C\pi_k y(\tau) + D\pi_k z(\tau) \tag{2.74}$$

$$\frac{d\bar{y}_k(t)}{dt} = A\bar{y}_k(t) + B\bar{z}_k(t) \tag{2.75}$$

$$\frac{d\pi_k y(\tau)}{d\tau} = A\pi_{k-1}y(\tau) + B\pi_{k-1}z(\tau) \tag{2.76}$$

Since we are considering the initial value problem, the matching conditions become

$$\pi_o z(0) + \bar{z}_o(0) = z^o \tag{2.77}$$

$$\pi_o y(0) + \bar{y}_o(0) = y^o \tag{2.78}$$

and for $k \geq 1$

$$\pi_k z(0) + \bar{z}_k(0) = 0 \tag{2.79}$$

$$\pi_k y(0) + \bar{y}_k(0) = 0 \tag{2.80}$$

and, due to our stability assumption,

$$\pi_k y(\infty) = \pi_k z(\infty) = 0 \quad k \geq 0 \tag{2.81}$$

Solutions of this type are referred to as "inner" and "outer", "fast" and "slow", or "steady state" and "boundary layer" depending on the author.

Our purpose here is to show that the series solution

$$\begin{aligned} y &= \bar{y} + \pi y \\ z &= \bar{z} + \pi z \end{aligned} \tag{2.82}$$

are equivalent to the solution of (2.3) obtained using the fundamental matrix (2.54). In other words,

$$\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} I \\ -P \end{bmatrix} e^{(A-BP)t} C_1 \tag{2.83}$$

$$\begin{bmatrix} \pi y \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \hat{P} \\ I \end{bmatrix} e^{(D+C\hat{P})t} C_2 \tag{2.84}$$

First, we seek an asymptotic solution to y_{slow} of the form

$$x(t) = x_o(t) + \mu x_1(t) + \dots\dots\dots\dots \tag{2.85}$$

and a solution of z_{slow} in the form

$$-Px(t) = -P_o x_o(t) - \mu(P_1 x_o + P_o x_1) \dots\dots\dots\dots$$

where x(t) is the transformation variable of (1.82). When (2.3) is used as the system model

$$\begin{bmatrix} \dot{x} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} A-P(\mu)B & 0 \\ 0 & \frac{D}{\mu} + BP(\mu) \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \tag{2.86}$$

Substituting in the formal power series (2.85) and (2.7) into (2.86) and equating like powers of μ, we obtain the following equations of the variations

$$\dot{x}_o = (A-BP_o)x_o \tag{2.87}$$

$$\dot{x}_1 = (A-BP_o)x_1 - BP_1 x_o \tag{2.88}$$

$$\vdots$$

$$\dot{x}_k = (A-BP_o)y_k - B\sum_{j=0}^{k-1} P_{k-j} x_j \tag{2.89}$$

We will now show that the differential equations necessary to solve for $\bar{y}_k$, $k \geq 0$ are equivalent to (2.89) $\forall k$. The equivalence of $\bar{z}_k$ and $(-Px)_k$ is a byproduct of the derivation for the equivalence of $\bar{y}_k$ and x_k.

From (2.65) and (2.67)

$$\bar{z}_o = - D^{-1}C\bar{y}_o \tag{2.90}$$

$$= - P_o\bar{y}_o$$

$$\frac{d\bar{y}_o}{dt} = (A-BP_o)\bar{y}_o \tag{2.91}$$

Now, from (2.69) and (2.71)

$$\bar{z}_1 = P_o\bar{y}_1 + D^{-1}\frac{d\bar{z}_o}{dt} \tag{2.92}$$

$$= - P_o\bar{y}_1 - D^{-2}C\frac{d\bar{y}_o}{dt}$$

$$= - P_o\bar{y}_1 - D^{-2}C(A-BD^{-1}C)\bar{y}_o$$

$$= - P_o\bar{y}_1 - P_1\bar{y}_o$$

$$\frac{d\bar{y}_1}{dt} = (A-BP_o)\bar{y}_1 - BP_1\bar{y}_o \tag{2.93}$$

and thus, for (2.73) and (2.75)

$$\bar{z}_k = - P_o y_k + D^{-1}\frac{dz_{k-1}}{dt}$$

$$\frac{d\bar{z}_{k-1}}{dt} = - P_o\frac{d\bar{y}_{k-1}}{dt} - P_1\frac{d\bar{y}_{k-2}}{dt} - \dots - P_{k-1}\frac{d\bar{y}_o}{dt} \tag{2.94}$$

$$\bar{z}_k = - P_o\bar{y}_k$$

$$- D^{-1}P_oA + D^{-1}P_oBP_o$$

$$- \ldots\ldots -$$

$$- D^{-1}P_{k-1}A + D^{-1} \sum_{j=0}^{k-1} P_jBP_{k-1-j}$$

$$= - P_o\bar{y}_k - P_1\bar{y}_{k-1} - \ldots\ldots - P_ky_o$$

$$= - \sum_{j=0}^{k} P_{k-j}\bar{y}_j \qquad (2.95)$$

$$\frac{d\bar{y}_k}{dt} = (A-BP_o)\bar{y}_k - B \sum_{j=0}^{k-1} P_{k-j}\bar{y}_j \qquad (2.96)$$

which is equivalent to (2.89) $\forall k$. Plus, it is obvious from (2.95) that $\bar{z}_k = -(Px)_k$, $\forall k$.

We now will show (2.84).

Using the dominant right eigenspace iterations, the singularly perturbed model (2.3) is transformed into

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\omega}} \end{bmatrix} = \begin{bmatrix} A-P(\mu)C & 0 \\ 0 & \frac{D+C\hat{P}(\mu)}{\mu} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\omega} \end{bmatrix} \qquad (2.97)$$

where
$$\hat{x}_o = y_o - \hat{P}z_o$$
$$\hat{\omega}_o = Qy_o + (I-\hat{Q}\hat{P})z_o$$

$\hat{P}$ is obtained from (2.78) and $\hat{Q}$ is then obtained from (2.90). The dichotomic nature of our fast and slow components led us to identify fast and slow components of y and z as

$$y_{slow} = (I-\hat{P}\hat{Q})\hat{x} \qquad y_{fast} = \hat{P}\hat{\omega}$$

$$z_{slow} = \hat{Q}\hat{x} \qquad z_{fast} = \hat{\omega}$$

The differential equation for the fast state vector is

$$\dot{\hat{\omega}} = \frac{D+C\hat{P}}{\mu} \tag{2.98}$$

Let $\tau = t/\mu$, then (2.98) becomes

$$\frac{d\hat{\omega}}{d\tau} = (D+C\hat{P})\omega(\tau) \tag{2.99}$$

We now seek an asymptotic solution to z_{fast} of the form

$$\omega(\tau) = \omega_o(\tau) + \mu\omega_1(\tau) + \ldots\ldots \tag{2.100}$$

and a solution to y_{fast} of the form

$$\hat{P}\omega(\tau) = P_o\omega_o(\tau) + \mu(P_o\omega_1(\tau) + P_1\omega_o(\tau)) + \ldots..$$

Substituting the formal power series (2.100) and into (2.29) and equating like powers of μ, we obtain the following equations of the variations

$$\frac{d\hat{\omega}_o(\tau)}{d\tau} = D\hat{\omega}_o(\tau) \tag{2.101}$$

$$\frac{d\hat{\omega}_1(\tau)}{d\tau} = D\hat{\omega}_1 + CP_1\omega_o \tag{2.102}$$

.
.
.
.

$$\frac{d\hat{\omega}_k}{d\tau} = D\hat{\omega}_k + C\sum_{j=0}^{k-1} \hat{P}_{k-j}\hat{\omega}_j \tag{2.103}$$

We will now show that the differential equations necessary to solve for $\pi_k z(\tau)$, $k \geq 0$ are equivalent to (2.103) k. The equivalence of $\pi_k y(\tau)$ and $(\hat{P}\hat{\omega})_k$ is a byproduct of this derivation.

From (2.68) and (2.81)

$$\pi_o y = 0 \tag{2.104}$$

Thus,

$$\frac{d\pi_o z}{d\tau} = D\pi_o z \tag{2.105}$$

From (2.72)

$$\frac{d\pi_1 y(\tau)}{d\tau} = A\pi_o y + B\pi_o z \tag{2.106}$$

Thus,

$$\pi_1 y(\tau) = \pi_1 y(0) + \int_o^\tau [A\pi_o y(\sigma) + B\pi_o z(\sigma)]d\sigma \tag{2.107}$$

To establish $\pi_1 y(0)$,

$$0 = \pi_1 y(0) + \int_o^\infty [A\pi_o y(\sigma) + B\pi_o z(\sigma) \tag{2.108}$$

Thus,

$$\pi_1 y(\tau) = -\int_\tau^\infty [A\pi_o y(\sigma) + B\pi_o z(\sigma)]d\sigma \tag{2.109}$$

Since
$$\pi_o y(\sigma) = 0$$

$$\pi_1 y(\tau) = -\int_\tau^\infty B\pi_o z(\sigma)d\sigma \tag{2.110}$$

$$= -\int_\tau^\infty BD^{-1} \frac{d\pi_o z(\sigma)d\sigma}{d\sigma}$$

$$= -BD^{-1}[\pi_o z(\infty) - \pi_o z(\tau)]$$

$$= BD^{-1}\pi_o z(\tau) = \hat{P}_1\pi_o z(\tau) \tag{2.111}$$

and as a result

$$\frac{d\pi_1 z(\tau)}{d\tau} = D\pi_1 z(\tau) + C\hat{P}_1\pi_o z(\tau) \tag{2.112}$$

Finally, for the k^{th} variation

$$\pi_k y(\tau) = -\int_\tau^\infty [A\pi_{k-1}y(\sigma) + B\pi_{k-1}a(\sigma)]d\sigma \tag{2.113}$$

$$= -BD^{-1}\int_\tau^\infty \frac{d\pi_{k-1}z(\sigma)d\sigma}{d\sigma}$$

$$-\int_\tau^\infty (A-BD^{-1}D)\hat{P}_1\pi_{k-2}z(\sigma)d\sigma$$

$$-\int_\tau^\infty (A-BD^{-1}C)\hat{P}_2\pi_{k-3}z(\sigma)d\sigma - \ldots . - \int_\tau^\infty (A-BD^{-1}C)\hat{P}_{k-1}\pi_o z(\sigma)d\sigma \tag{2.114}$$

using

$$\pi_{k-1}(\tau) = D^{-1}\frac{d\pi_{k-1}z(\tau)}{d\dagger} - D^{-1}C\sum_{j=0}^{k-2}\hat{P}_{k-1-j}\pi_j z(\tau) \tag{2.115}$$

$$\pi_k y(\tau) = -\hat{P}_1 \int_\tau^\infty \frac{d\pi_{k-1} z(\sigma)}{d\sigma}\, d\sigma - \hat{P}_2 \int_\tau^\infty \frac{d\pi_{k-2} z(\sigma)}{d\sigma}\, d\sigma -$$

$$\ldots\ldots - \hat{P}_k \int_\tau^\infty \frac{d\pi_o z(\sigma)}{d\sigma}\, d\sigma$$

$$= \hat{P}_1 \pi_{k-1} z(\tau) + \hat{P}_2 \pi_{k-2} z(\tau) + \ldots\ldots + \hat{P}_k \pi_o z(\tau) \tag{2.116}$$

$$= \sum_{j=0}^{k-1} \hat{P}_{k-j} \pi_j z(\tau) \tag{2.117}$$

Thus,
$$\frac{d\pi_k z(\tau)}{d\tau} = D\pi_k z(\tau) + C \sum_{j=0}^{k-1} \hat{P}_{k-j} \pi_j z(\tau) \tag{2.118}$$

Which is equivalent to (2.103) $\forall k \geq 0$. Also, from (2.117),

$$(\pi_y)_k = (\hat{P}\hat{\omega})_k \quad \forall k \geq 0 \tag{2.119}$$

Thus, we have shown that $\bar{y}_o$ satisfies

$$\frac{dy_o}{dt} = (A - BP)\bar{y}_o \tag{2.120}$$

and that

$$\frac{d\pi_z(\tau)}{d\tau} = (D + C\hat{P})\pi_z(\tau) \tag{2.121}$$

likewise

$$\begin{aligned} \pi_y &= \hat{P}\pi_z(\tau) \\ \bar{z}_o &= -P\bar{y}_o(t) \end{aligned} \tag{2.122}$$

The matching conditions (2.77-80) thus reduce to

$$y_o = \bar{y}(0) + \pi y(0) \tag{2.123}$$

$$z_o = \bar{z}(0) + \pi z(0) \tag{2.124}$$

However, from (2.117) and 2.95)

$$y_o = \bar{y}(0) + \hat{P}\pi_z(0) \tag{2.125}$$

$$z_o = - P\bar{y}(0) + \pi z(0) \tag{2.126}$$

or

$$\begin{bmatrix} y_o \\ z_o \end{bmatrix} = \begin{bmatrix} I & \hat{P} \\ -P & I \end{bmatrix} \begin{bmatrix} y(0) \\ \pi_z(0) \end{bmatrix} \tag{2.127}$$

Thus,

$$\begin{bmatrix} y(0) \\ \pi_z(0) \end{bmatrix} = \begin{bmatrix} I & \hat{P} \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} y_o \\ z_o \end{bmatrix} \tag{2.128}$$

Which uniquely determines the solutions to (1.20) and (1.21) and then (2.83) and (2.84). However, given the fundamental set (2.54) and the initial conditions $y(0) = y_o$, $z(0) = z_o$, the solution to the initial value problem is uniquely determined by

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^{(A-BP)t} & \hat{P}e^{(D+PB)t} \\ -Pe^{(A-BP)t} & e^{(D+C\hat{P})t} \end{bmatrix} \begin{bmatrix} I & \hat{P} \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} y_o \\ z_o \end{bmatrix} \tag{2.129}$$

In conclusion, given the initial value problem (2.3) that satisfies condition (1.21), the presence of the singular perturbation

parameter μ suggests seeking a series solution in two-time scales. The time scale t corresponds to the slow or "outer" solutions. The time scale $\tau = t/\mu$ corresponds to the fast or "inner" solution which effectively blows up the initial region of nonuniformity. Vasil'eva's method of matched asymptotic expansions is an asymptotic method of decomposing the solution vector of (2.3) into fast and slow components. The mechanism for separation is based on the functions f and F dependence on the independent variables t and τ. What we have shown here is that for linear time-invariant singularly perturbed systems, the explicit time-scale decomposition of matched asymptotic expansions is equivalent to the eigenstructure decompositions of Chapter 1.

E. Simplified Iterative Schemes

One of the computational drawbacks of the dominant left and right eigenspace iterations is the computation of the inverses

$$(D + \mu P_k B)^{-1} \tag{2.130}$$

$$(D - C\hat{P}_k)^{-1} \tag{2.131}$$

at every iteration.

Looking at the case (2.6), the iterative matrix recursion is

$$P_{k+1} = (D + \mu P_k B)^{-1} \cdot (C + \mu P_k A) \tag{2.132}$$

which can be expressed as

$$DP_{k+1} = - \mu P_k B P_{k+1} + C + \mu P_k A \tag{2.133}$$

if this is approximated by

$$D\tilde{P}_{k+1} = - \mu \tilde{P}_k BP_k + C + \mu \tilde{P}_k A \tag{2.134}$$

$$\tilde{P}_{k+1} = D^{-1}(C + \mu \tilde{P}_k A - \mu \tilde{P}_k B\tilde{P}_k) \tag{2.135}$$

will have eliminated the need for the inverse (2.130) at every iteration. However, the question remains as to whether (1.135) converges or not and if it does, does it converge to the dichotomic solution P. To answer these questions, let us take an in depth look at the "equations of variation" for (1.132). In Section B, we obtained an asymptotic expansion in μ to the solution of (1.132). We did this by equating like powers of μ in the equation

$$[D+\mu(P_k^o+\mu P_k^1+\ldots\ldots\ldots)B](P_{k+1}^o+\mu P_{k+1}^1+\ldots\ldots) \qquad (2.136)$$

$$= C+\mu(P_k^o+\mu P_k^1+\ldots\ldots)A$$

and obtaining the equations of variation

$$DP_{k+1}^o = C \qquad P_o^o = D^{-1}C \qquad (2.137)$$

$$DP_{k+1}^1 + P_k^o BP_{k+1}^o = P_k^o A \qquad P_o^1 = 0$$

$$DP_{k+1}^2 + P_k^o BP_{k+1}^1 + P_k^1 BP_{k+1}^o = P_k^1 A \qquad P_o^2 = 0$$

$$\vdots$$

$$DP_{k+1}^N + \sum_{j=0}^{N-1} P_k^j BP_{k+1}^{N-1-j} = P_k^{N-1}A \qquad P_o^N = 0 \qquad (2.138)$$

if we plot the solutions of the equations (2.138), we obtain the results of Figure 2.1. In other words, the N^{th} variation does not reach its equilibrium value until $k = N$.

Thus, $\forall N \geq 0$

$$P = \sum_{j=0}^{N} P^j\mu^j + 0(\mu^N) \qquad (2.139)$$

where

$$P^o = D^{-1}C$$

$$P^1 = D^{-1}P^oA - D^{-1}P^oBP^o$$

$$\vdots$$

$$P^N = D^{-1}P^{N-1}A - D^{-1}\sum_{j=0}^{N-1} P^jBP^{N-1-j} \qquad (2.140)$$

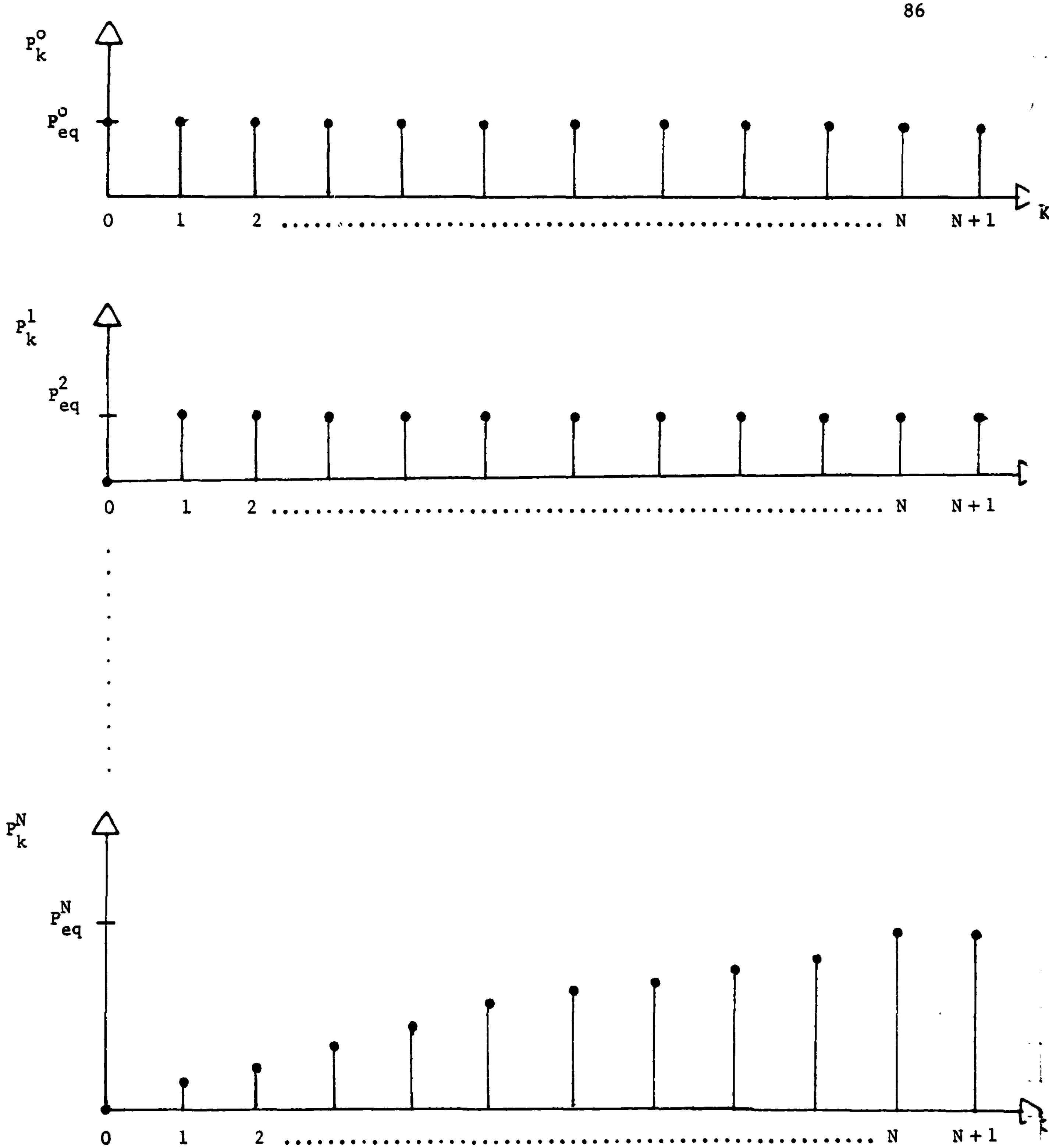

Figure 2.1. Variational Equations for Iterative Scheme (2.132).

In Section B of this chapter we established bounds on μ such that (2.139) is a convergent MacLauren series.

Now, let us obtain an asymptotic scales solution to (2.135) and compare it to (2.139). Substituting in a formal series for $\tilde{P}_k$, we obtain

$$(\tilde{P}^o_{k+1} + \mu\tilde{P}^1_{k+1} + \ldots.) = D^{-1}[C + \mu(\tilde{P}^o_k + \mu\tilde{P}^1_k + \ldots)A$$
$$- \mu(\tilde{P}^o_k + \mu\tilde{P}^1_k + \ldots)B(\tilde{P}^o_k + \mu\tilde{P}^1_k + \ldots)] \qquad (2.141)$$

which gives us the following variational equations

$$\tilde{P}^o_{k+1} = D^{-1}C \qquad (2.142)$$

$$\tilde{P}^1_{k+1} = D^{-1}\tilde{P}^o_k A - D^{-1}\tilde{P}^o_k B\tilde{P}^o_k$$

$$\vdots$$

$$\tilde{P}^N_{k+1} = D^{-1}\tilde{P}^{N-1}_k A - D^{-1}\sum_{j=0}^{N-1}\tilde{P}^j_k B\tilde{P}^{N-1-j}_k \qquad (2.143)$$

If we plot the solutions of these equations we obtain the results of Figure 2.2.

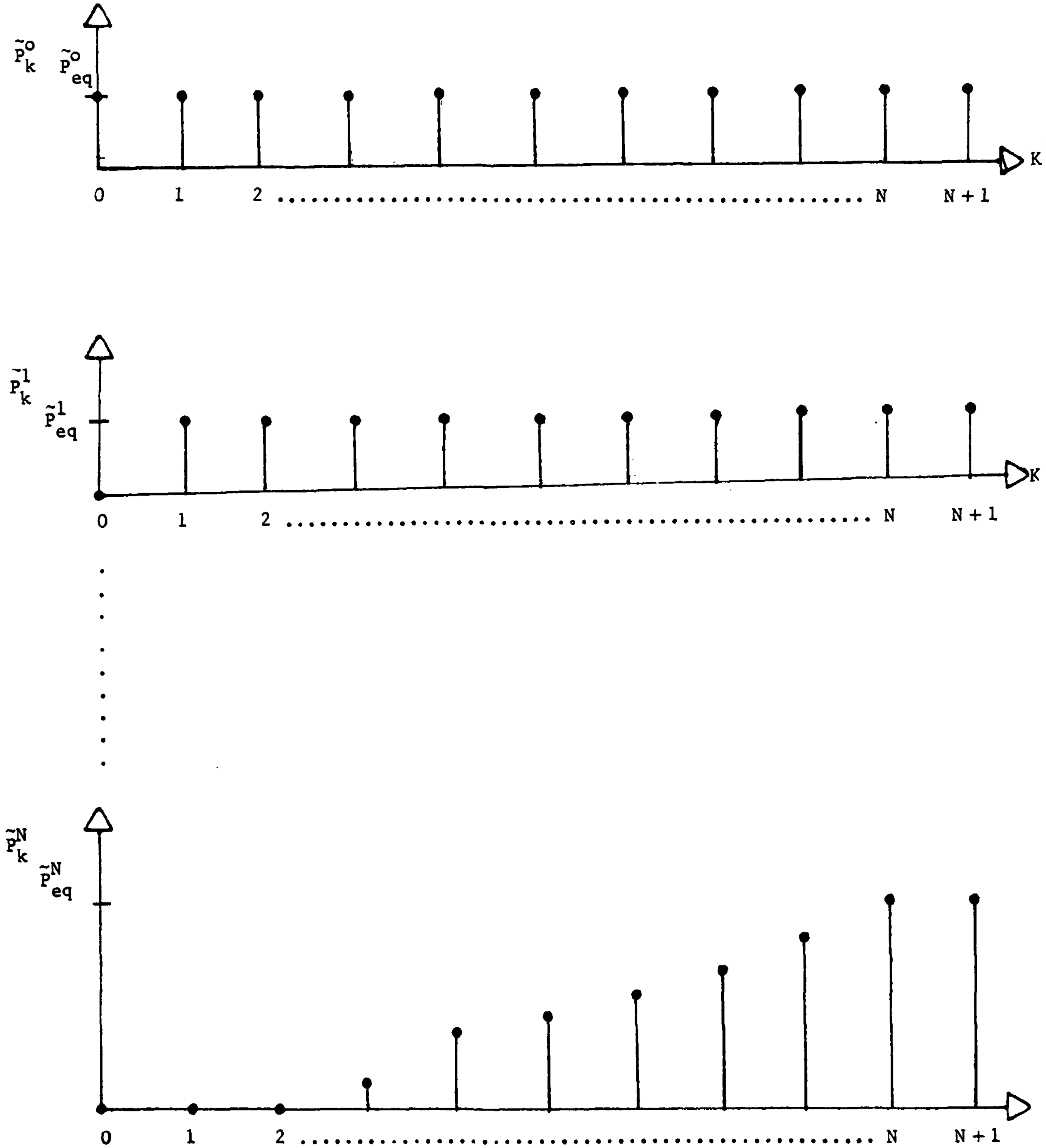

Figure 2.2. Variational Equations for Simplified Iterative Scheme (2.135).

Again, the N^{th} variation does not reach its equilibrium value until $K = N$. Moveover, $\forall N \geq 0$

$$\tilde{P} = \sum_{j=0}^{N} P^j \mu^j + 0(\mu^2) \tag{2.144}$$

where

$$\tilde{P}^o = D^{-1}C$$

$$\tilde{P}^1 = D^{-1}P^oA - D^1\tilde{P}^oB\tilde{P}^o$$

$$\vdots$$

$$\tilde{P}^N = D^{-1}\tilde{P}^{N-1}A - D^{-1}\sum_{j=0}^{N-1} \tilde{P}^j B \tilde{P}^{N-1-j} \tag{2.145}$$

which are equivalent to (2.139) $\forall N \geq 0$*.* However, we know that if

$$0 \leq \mu < \frac{1}{\|D^{-1}\| \, \|B\| M} \tag{2.146}$$

Then (2.139) is a convergent MacLauren series. Thus, for μ satisfying (2.146), (2.148) is bounded and converges to the dichotomic solution P.

Obtaining bounds for the convergence of (2.135) to the dichtomic equilibrium were first established in [4]. In this case a contraction mapping argument is used to show convergence if

$$0 \leq \mu < \frac{1}{3\|D^{-1}\|(\|A_o\| + \|B\| \, \|D^{-1}C\|} \tag{2.147}$$

where $A_o = A - BD^{-1}C$

To show one improvement over this bound, we note that from [4]

$$\|P\| \le \|D^{-1}C\| + \frac{2\|A_o\|\,\|D^{-1}C\|}{\|A_o\| + \|B\|\,\|D^{-1}C\|} \tag{2.148}$$

$$\le \frac{3\|A_o\|\,\|D^{-1}C\| + \|B\|\,\|D^{-1}C\|^2}{\|A_o\| + \|B\|\,\|D^{-1}C\|} \tag{2.149}$$

thus bound (2.146) becomes

$$0 \le \mu \frac{1}{\|D^{-1}\|\,\|B\|} \cdot \frac{\|A_o\| + \|B\|\,\|D^{-1}C\|}{(3\|A_o\|\,\|D^{-1}C\| + \|B\|\,\|D^{-1}C\|^2)}$$

$$< \frac{\|A_o\| + \|B\|\,\|D^{-1}C\|}{\|D^{-1}\|\,\|B\|\,\|D^{-1}C\|(3\|A_o\| + \|B\|\,\|D^{-1}C\|)} \tag{2.150}$$

which is obviously less conservative than (2.147). The dual to this result involving equation

$$\hat{P}_{k+1} = \mu(B + A\hat{P}_k)(D + C\hat{P}_k)^{-1} \tag{2.151}$$

is derived in the analogus manner. The simplified interation in this case becomes

$$\hat{\tilde{P}}_{k+1} = (\mu B + \mu A\hat{\tilde{P}}_k - \hat{\tilde{P}}_k C\hat{\tilde{P}}_k)D^{-1} \tag{2.152}$$

$$\hat{\tilde{P}}_o = \mu BD^{-1}$$

and converge to the dichotomic solution $\hat{P}$ when μ is bounded by (2.33).

In the next section we give an example highlighting many of the important results in this chapter.

F. Example - Fundamental Solution Sets

Given the same power system model from our previous example given here in singularly perturbed form

$$\begin{bmatrix} \dot{y} \\ \hline \mu\dot{z} \end{bmatrix} = \begin{bmatrix} -.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.167 & 0 & 0 & 0 & 0 & 0 & .167 \\ 0 & 0 & -.5 & 0 & .2 & 0 & 0 & 0 \\ 0 & .009 & 0 & -.112 & -.063 & .014 & .116 & .01 \\ 0 & -.075 & 10.0 & -9.101 & -3.994 & -.112 & -.927 & -.08 \\ 0 & 2.0 & 0 & 0 & 0 & -2.0 & 0 & 0 \\ 0 & 0 & 0 & -.278 & 1.319 & 0 & -1.389 & 0 \\ 4.75 & 0 & 0 & 0 & 0 & 0 & 0 & -5.0 \end{bmatrix} \begin{bmatrix} y \\ \hline z \end{bmatrix} \tag{2.153}$$

We seek a solution of the form

$$\dot{x} = \begin{bmatrix} e^{(A-BP)t} & \hat{P}e^{(D+C\hat{P})t} \\ -Pe^{(A-BP)t} & e^{(D+C\hat{P})t} \end{bmatrix} \cdot \begin{bmatrix} I & \hat{P} \\ -P & I \end{bmatrix}^{-1} \cdot \begin{bmatrix} x_o \end{bmatrix} \tag{2.154}$$

using the reduced order iterations of Section E.

To use (2.135) and (2.154) we must first examine the bounds on μ that guarantee convergence.

For (2.135),

$$0 \le \mu < 3.56285 \tag{2.155}$$

and for (2.152)

$$0 \le \mu < 4.42141 \tag{2.156}$$

Since $\mu = 1$ in our problem (2.153), we are well within bounds.

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Using $\tilde{P}_o$

$$A - \tilde{P}_o B = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.15834 & -0.16667 & 0.00000 & 0.00000 \\ -0.00312 & -0.00766 & -0.08981 & -0.36571 \\ 0.00877 & 0.02153 & 0.09635 & -0.22145 \end{bmatrix} \quad (2.157)$$

with eigenvalues

$$\begin{matrix} -0.15563 & +0.17579\ J \\ -0.15563 & -9.17579\ J \\ -0.16667 & +0.00000\ J \\ -0.20000 & +0.00000\ J \end{matrix} \quad (2.158)$$

Likewise, using $\hat{\tilde{P}}$

$$D + \hat{\tilde{P}}_o C = \begin{bmatrix} -4.31691 & -0.03023 & 0.04757 & -0.05415 \\ 0.00000 & -2.00000 & 0.00000 & -0.06667 \\ 1.32208 & 0.00179 & -1.36749 & 0.00051 \\ 0.00000 & 0.00000 & 0.00000 & -5.00000 \end{bmatrix} \quad (2.159)$$

with eigenvalues

$$\begin{matrix} -4.33808 & +0.00000\ J \\ -1.34632 & +0.00000\ J \\ -2.00000 & +0.00000\ J \\ -5.00000 & +0.00000\ J \end{matrix} \quad (2.160)$$

Using an initial condition of

$$x_o^t = (1,\ 2,\ 3,\ -2,\ 1,\ -1,\ 4,\ 2)$$

The full accuracy original states of (2.153) are plotted versus their 0th order approximations using (2.145) on the next several pages. Some higher order plots are also given where convergence verification is needed.

Actual State ________________

Approximated ------------------

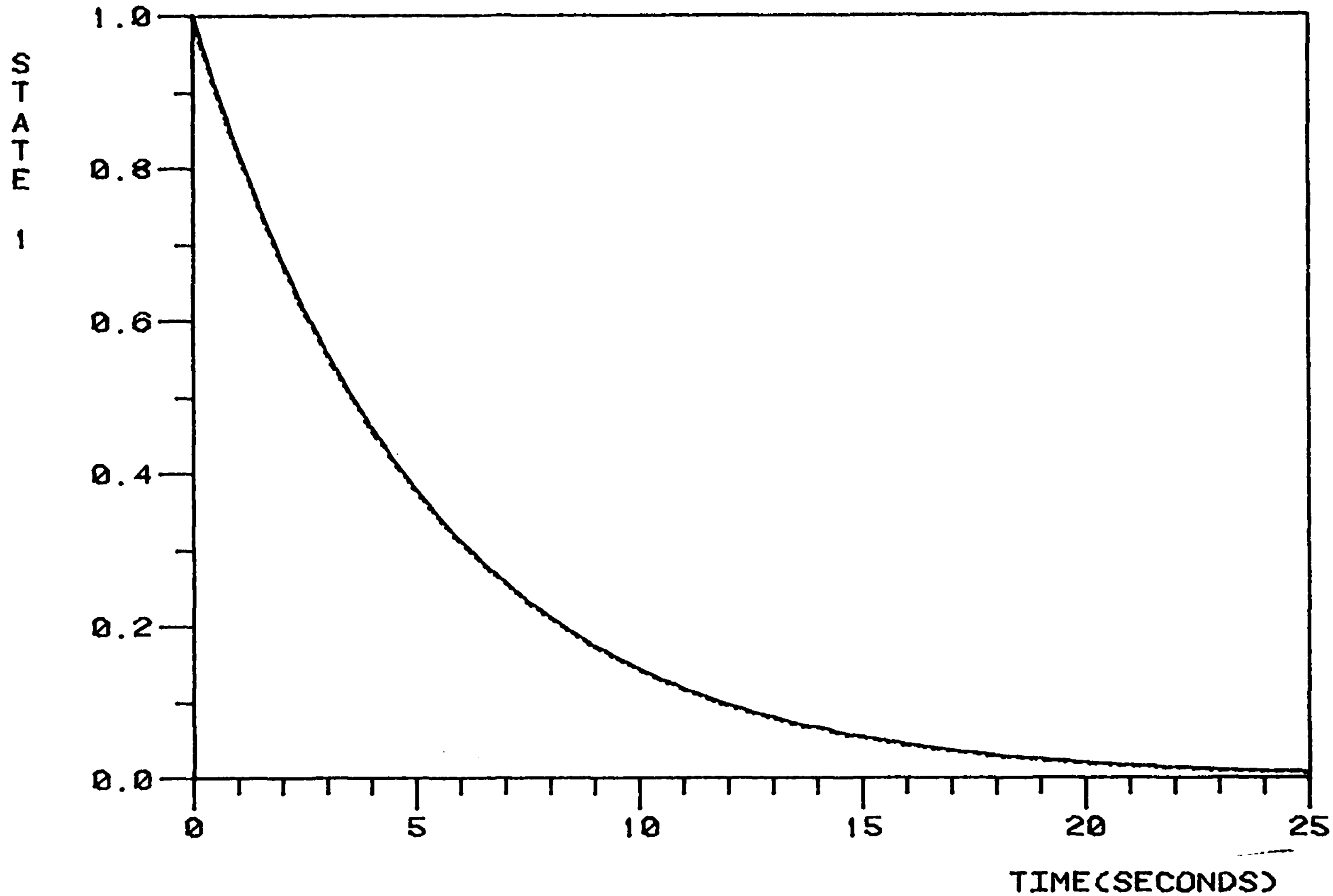

Figure 2.3. State 1 0^{th} order approximation.

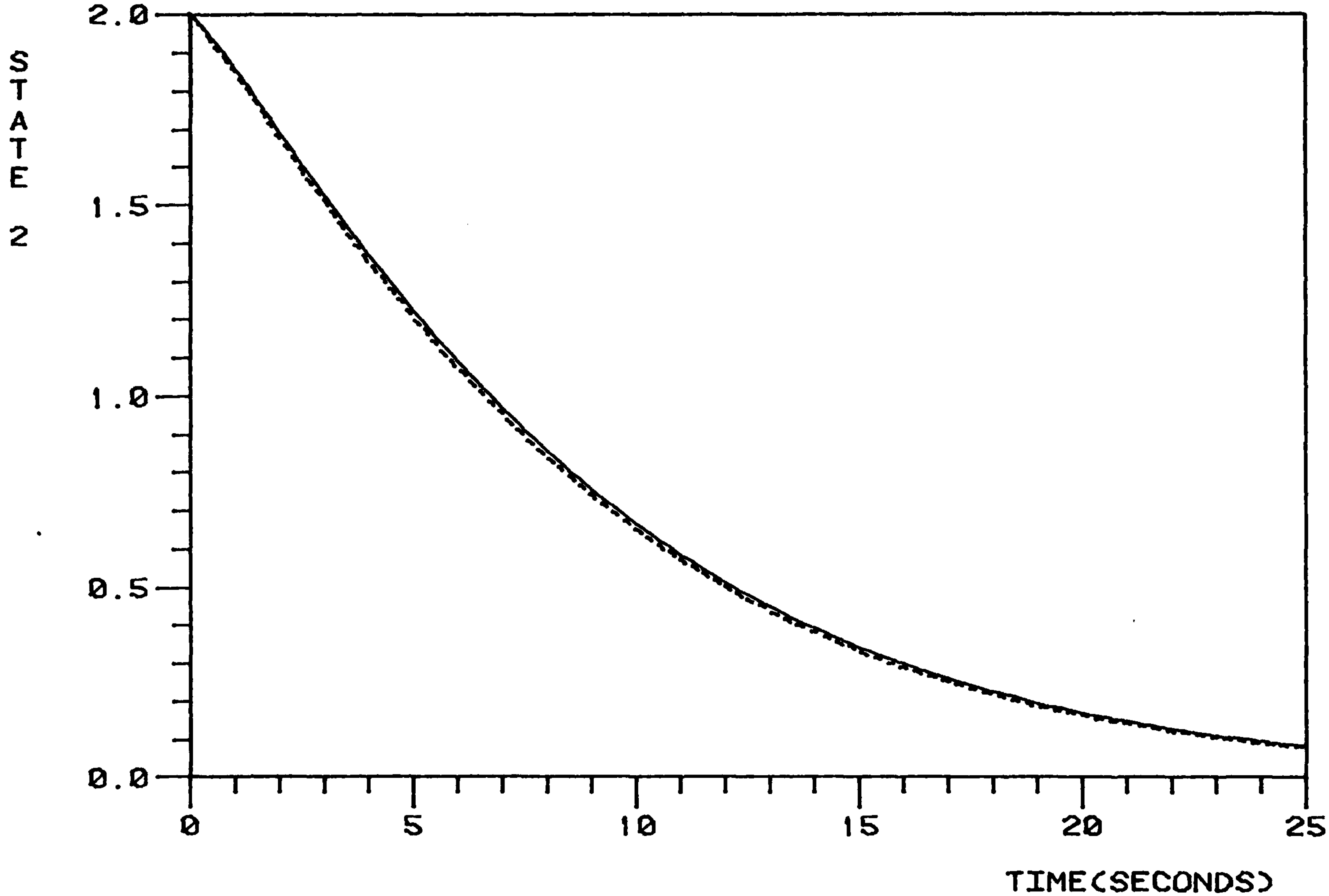

Figure 2.4. State 2 0^{th} order approximation.

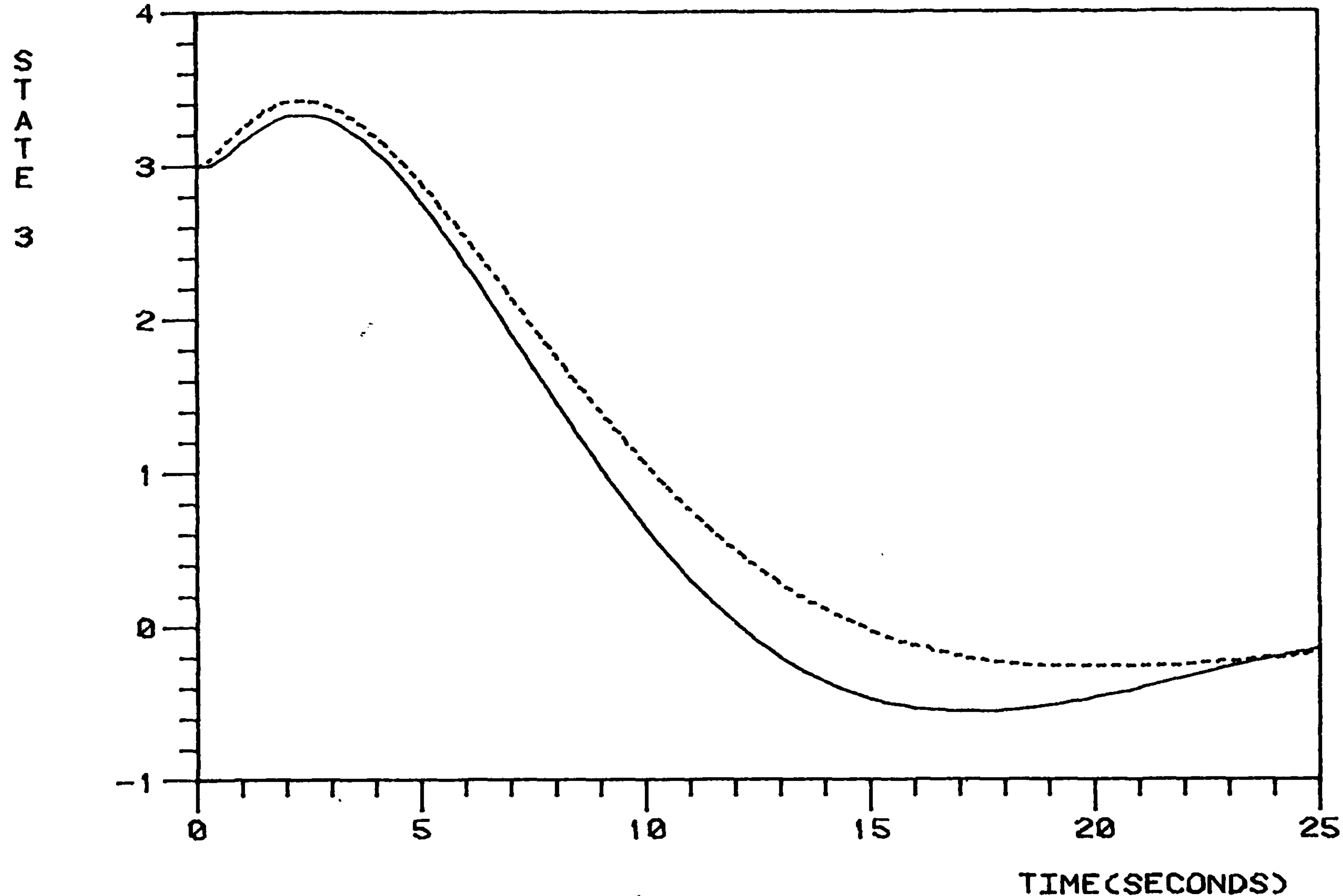

Figure 2.5. State 3 0^{th} order approximation

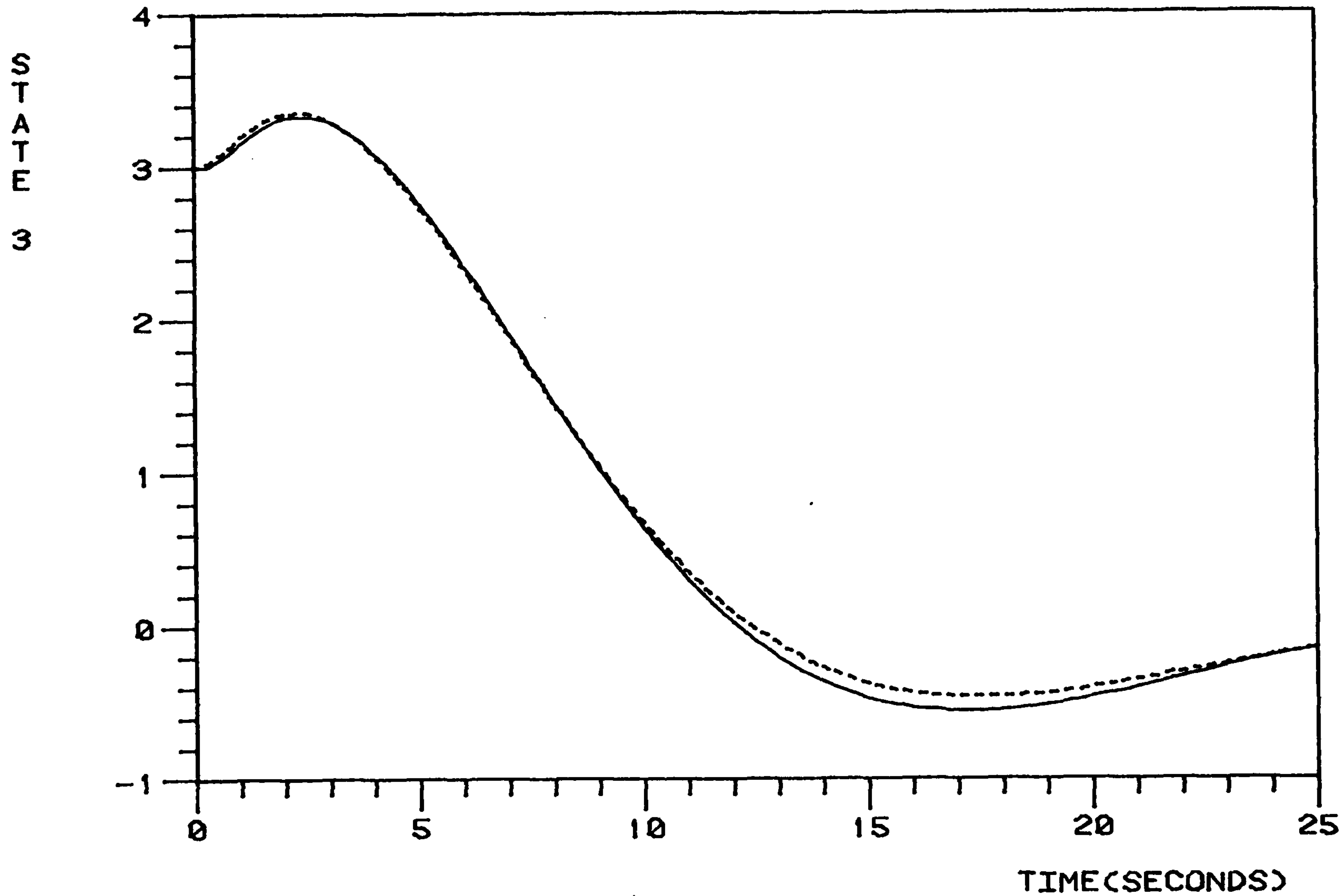

Figure 2.6. State 3 1st order approximation.

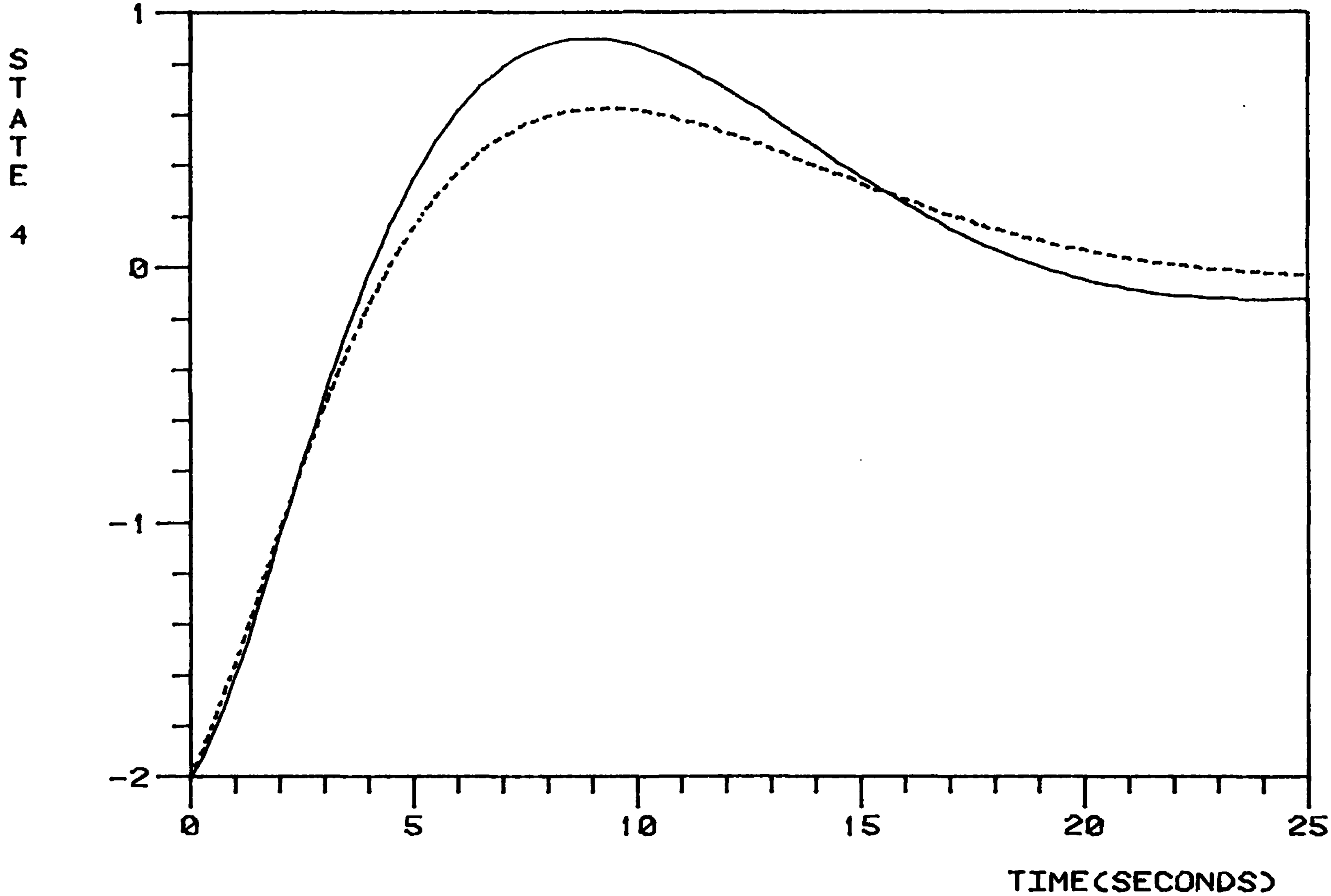

Figure 2.7. State 4 0^{th} order approximation.

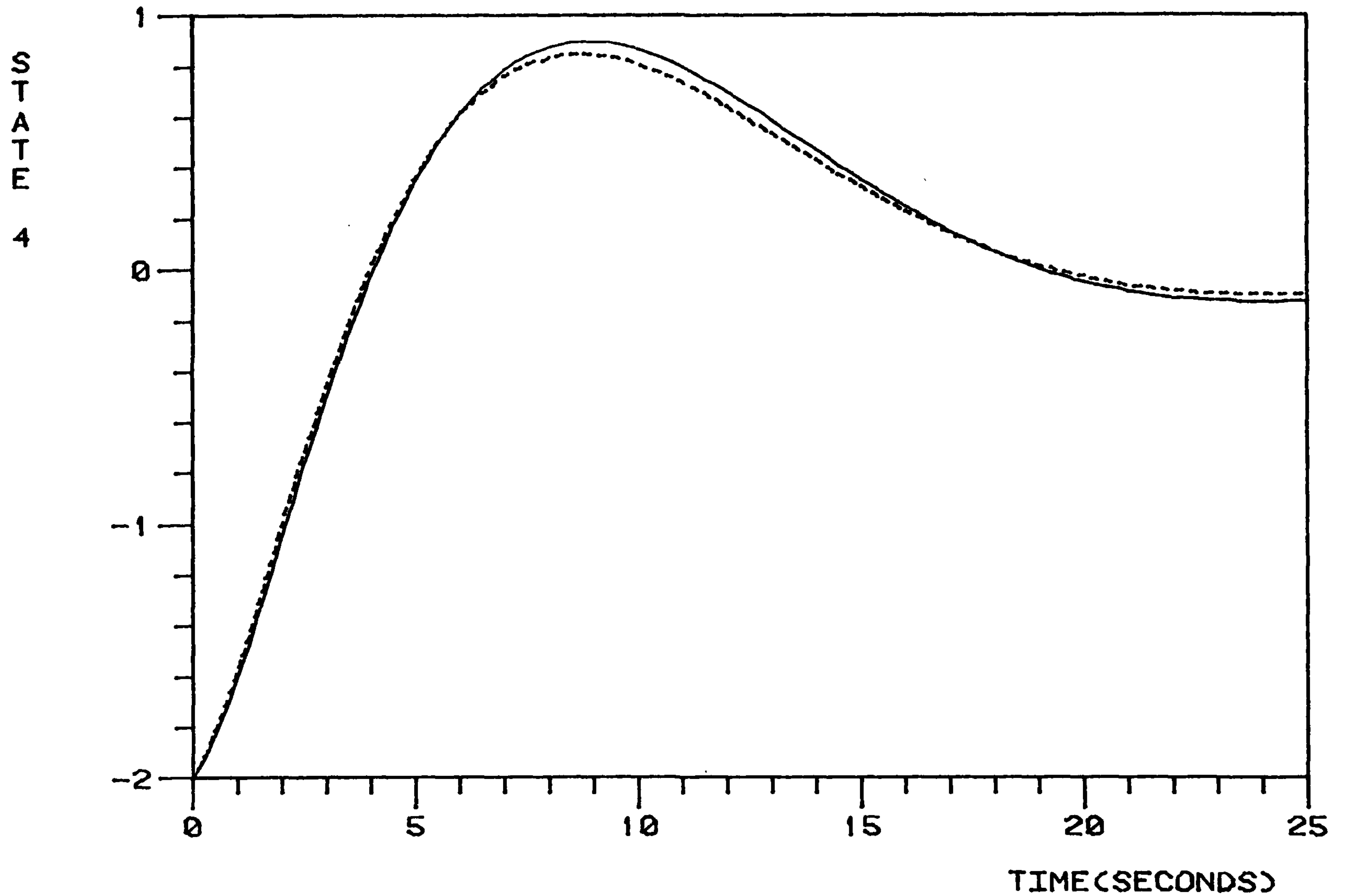

Figure 2.8. State 4 1st order approximation.

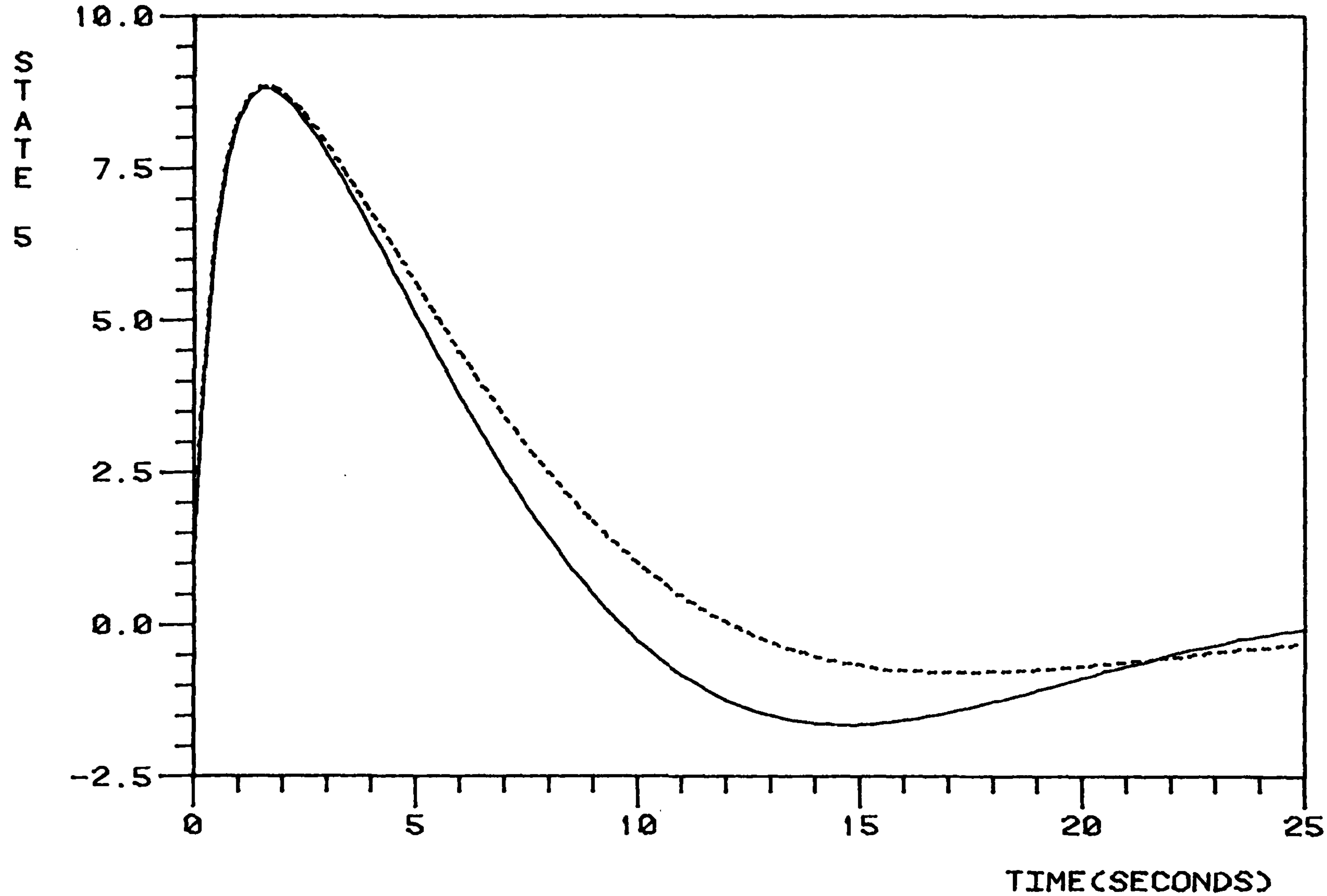

Figure 2.9. State 5 0^{th} order approximation.

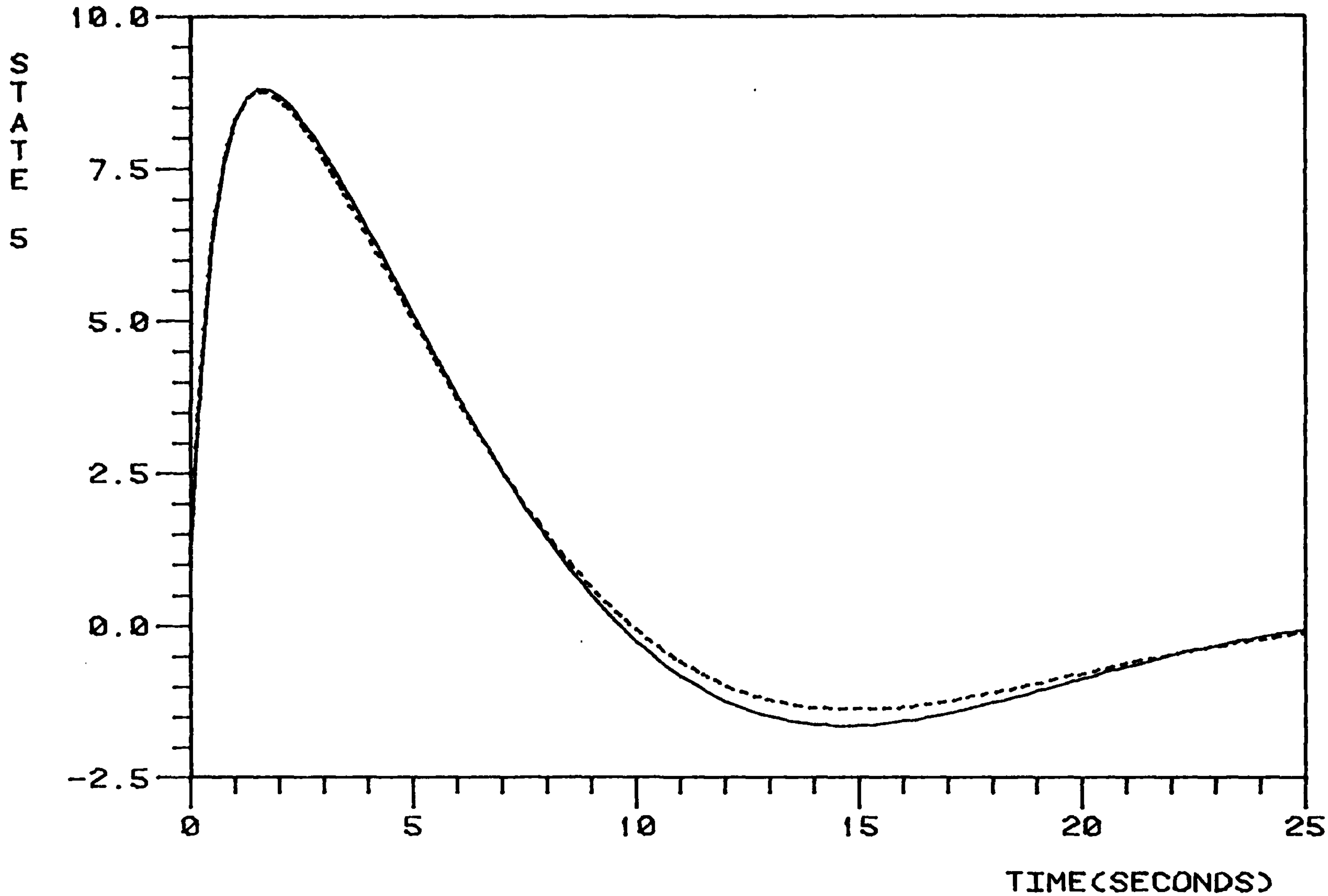

Figure 2.10. State 5 1st order approximation.

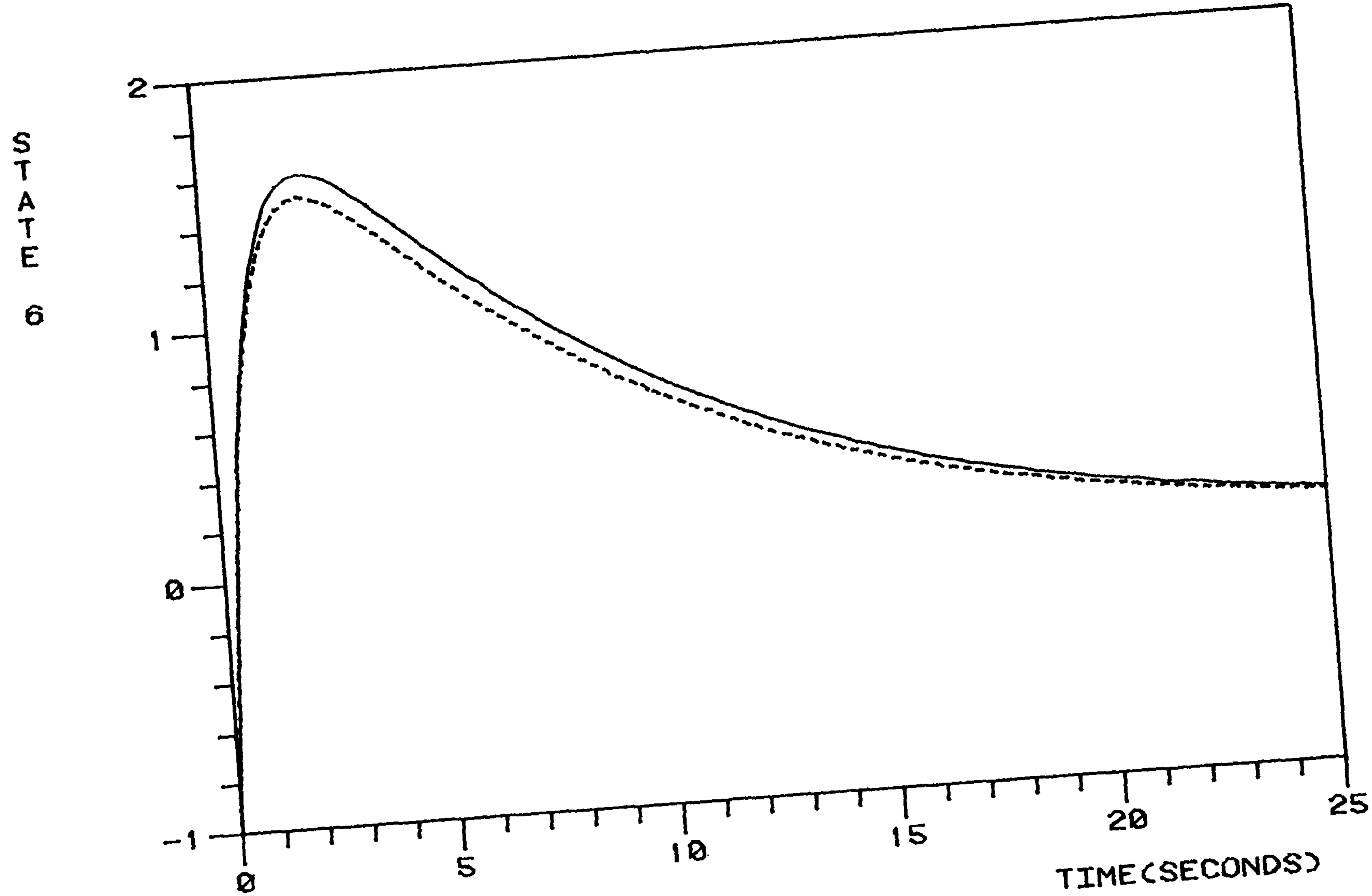

Figure 2.11. State 6 0^{th} order approximation.

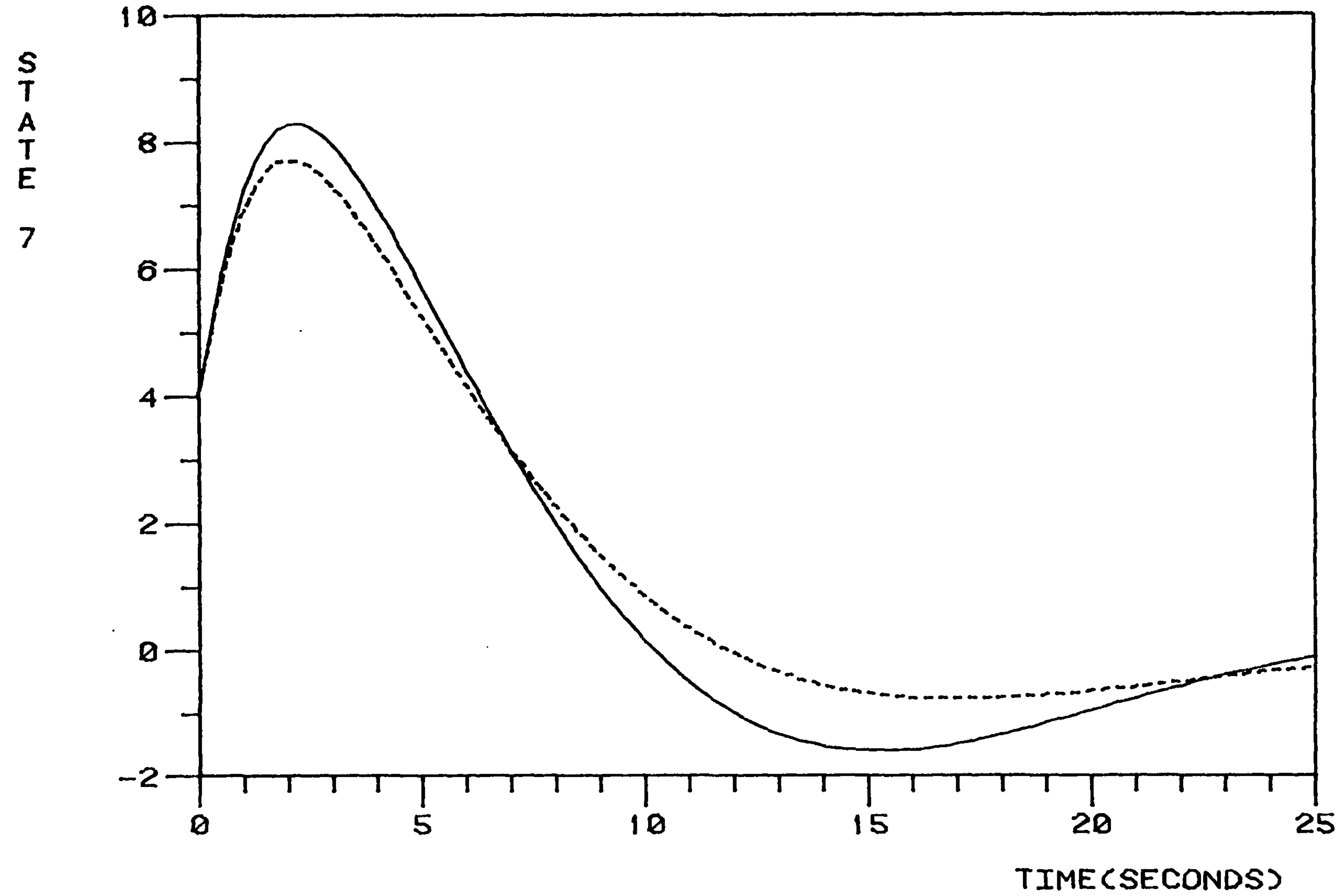

Figure 2.12. State 7 1st order approximation.

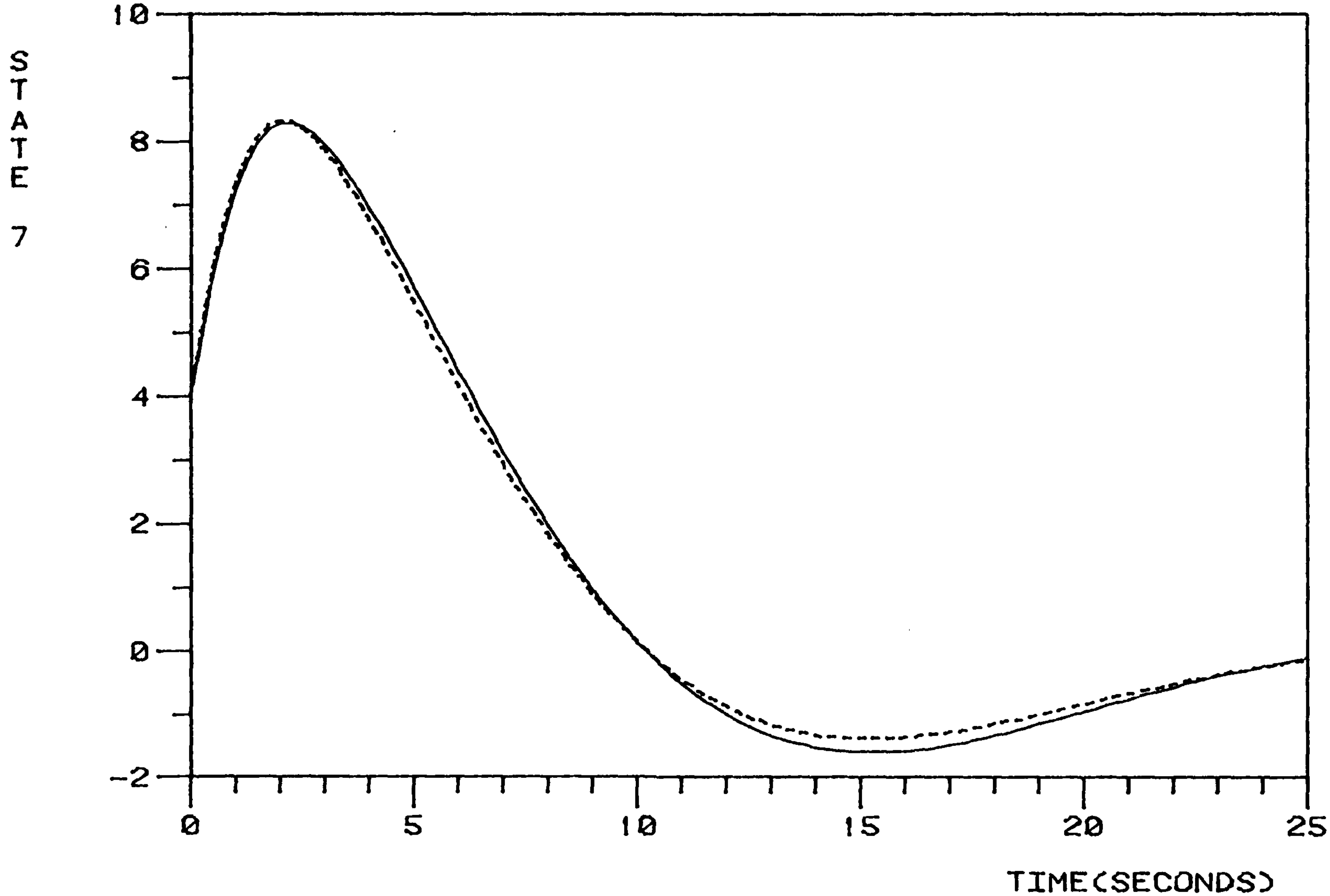

Figure 2.13. State 7 1^{st} order approximation.

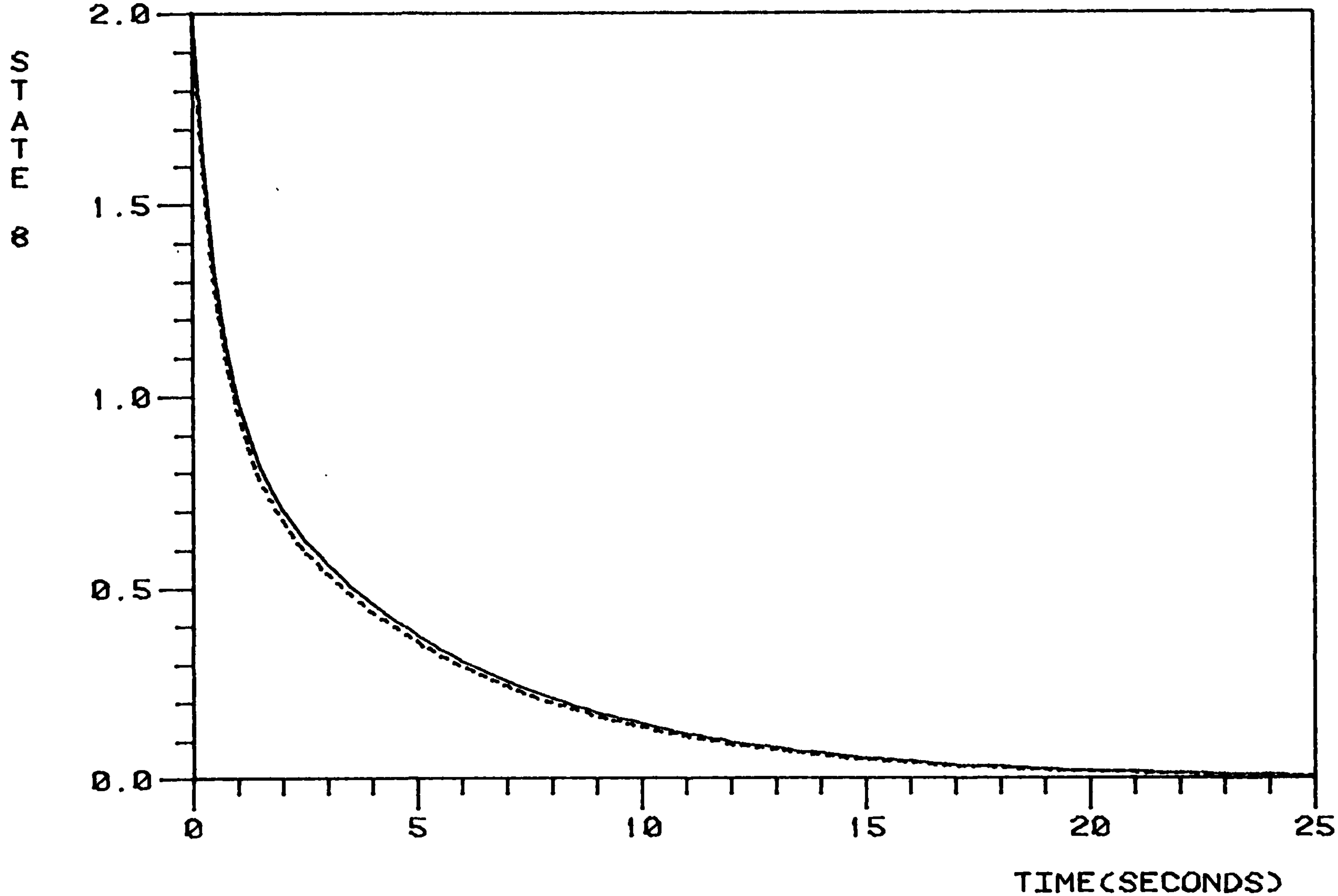

Figure 2.14. State 8 0^{th} order approximation.

G. Example - Reduced Order Iterations

In the previous example $\mu = 1$ easily satisfied the bounds (2.155) and (2.156). In this example we will allow $\mu = 3$ and we will examine the convergence. It is important to note that previous bounds given in [4] are for (2.135)

$$0 \le \mu < 1.2216 \tag{2.161}$$

and for (2.152)

$$0 \le \mu < 1.4082 \tag{2.162}$$

Thus, allowing $\mu = 3$ is an appropriate test for our new bounds. With $\mu = 3$, the eigenvalues of (2.153) are

$$\begin{array}{ll} -0.4545519 & +0.0000000\ J \\ -0.0328825 & +0.2174586\ J \\ -0.0328825 & -0.2174586\ J \\ -1.8864767 & +0.0000000\ J \\ -0.6666667 & +0.0000000\ J \\ -0.1666700 & +0.0000000\ J \\ -1.6666667 & +0.0000000\ J \\ -0.2000000 & +0.0000000\ J \end{array} \tag{2.163}$$

using $\tilde{P}_o$

$$A - B\tilde{P}_o = \begin{bmatrix} -0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.15834 & -0.16667 & 0.00000 & 0.00000 \\ -0.00312 & -0.00766 & -0.08981 & -0.36571 \\ 0.00788 & 0.02153 & 0.09635 & -0.22145 \end{bmatrix} \tag{2.164}$$

with eigenvalues

$$\begin{array}{ll} -0.15563 & +0.17579\ J \\ -0.15563 & -0.17579\ J \\ -0.16667 & +0.00000\ J \\ -0.20000 & +0.00000\ J \end{array} \tag{2.165}$$

Likewise, using $\hat{\tilde{P}}_o$

$$D + C\hat{\tilde{P}}_o = \begin{bmatrix} -1.65397 & 0.04443 & 0.66609 & -0.00082 \\ 0.00000 & -0.66667 & 0.00000 & -0.06667 \\ 0.44248 & 0.00179 & -0.44156 & 0.00051 \\ 0.00000 & 0.00000 & 0.00000 & -1.66667 \end{bmatrix} \qquad (2.166)$$

with eigenvalues

$$\begin{array}{ll} -1.86153 & +0.00000\ J \\ -0.23400 & +0.00000\ J \\ -0.66667 & +0.00000\ J \\ -1.66667 & +0.00000\ J \end{array} \qquad (2.167)$$

Some eigenvalues are very accurate while some have a substantial error. Using $\tilde{P}_5$

$$A - B\tilde{P}_5 = \begin{bmatrix} 0.20000 & 0.00000 & 0.00000 & 0.00000 \\ 0.17993 & -0.16667 & 0.00000 & 0.00000 \\ -0.00090 & -0.00270 & -0.05903 & -0.32775 \\ -0.00053 & 0.02545 & 0.11841 & -0.02838 \end{bmatrix} \qquad (2.168)$$

with eigenvalues

$$\begin{array}{ll} -0.04371 & +0.19641\ J \\ -0.04371 & -0.19641\ J \\ -0.16667 & +0.00000\ J \\ -0.20000 & +0.00000\ J \end{array} \qquad (2.169)$$

likewise, using $\hat{\hat{P}}_5$

$$D + C\hat{\hat{P}}_5 = \begin{bmatrix} -1.90740 & 0.00077 & -0.11943 & -0.00521 \\ 0.00000 & -0.66667 & 0.00000 & -0.07408 \\ 0.44430 & 0.00253 & -0.43118 & 0.00040 \\ 0.00000 & 0.00000 & 0.00000 & -1.66667 \end{bmatrix} \qquad (2.170)$$

with eigenvalues

$$\begin{array}{ll} -1.87053 & +0.00000\ J \\ -0.46805 & +0.00000\ J \\ -0.66667 & +0.00000\ J \\ -1.66667 & +0.00000\ J \end{array} \qquad (2.171)$$

Continuing, the eigenvalues of $A - B\hat{\hat{P}}_{15}$ are

$$
\begin{array}{rr}
-0.03291 & +0.21694\ J \\
-0.03291 & -0.21694\ J \\
-0.16667 & +0.00000\ J \\
-0.20000 & +0.00000\ J
\end{array}
\qquad (2.172)
$$

and the eigenvalues of $D + C\hat{\tilde{P}}_{15}$ are

$$
\begin{array}{rr}
-1.88648 & +0.00000\ J \\
-0.45457 & +0.00000\ J \\
-0.66667 & +0.00000\ J \\
-1.66667 & +0.00000\ J
\end{array}
$$

Thus, we have convergence. However, the rate of convergence decreases as μ approaches the circle of convergence.

III. CONCLUSIONS

In this report we have presented various methods used decomposing large scale systems into reduced order subsystems. These methods came under such names as quasi steady state, Riccati-iterates, and matched asymptotic expansions in previous works. Using power iterations for obtaining the left and right dominant eigenspace of a matrix we have been able to show that the convergent behavior of the above methods to be equivalent. Decomposition of time scales in linear systems reduces to separation of invariant eigenspaces. If there is a magnitude separation in eigenvalues, then it is possible, using the above mentioned procedures, to iteratively solve for the invariant eigenspaces and generate reduced order models.

In Chapter II of this report we have shown that a linear singularly perturbed system that is decomposed through matched asymptotic expansions is in essence an eigenspace decomposition. The presence of the singular perturbation parameter μ enables us to obtain convergent power series solutions to the block diagonalization matrices P and $\hat{P}$ that enabled us to obtain computationally more efficient algorithms for obtaining the time scale decompositions. The singularly perturbed model can thus be looked upon as an application of parameter imbedding in an effort to obtain a power series decomposition in two time scales, namely t and $\tau = t/\mu$.

Finally, in the Appendix, we give an application of this work involving partial pole placement.

APPENDIX - APPLICATIONS TO PARTIAL POLE PLACEMENT

There are many applications using the techniques developed in the first two chapters. They include robust designs, reduced order regulator designs, and reduced order modeling only to mention a few. In this appendix we will show the time scale decomposition techniques can be used to implement partial or full pole placement design on both continuous and discrete systems.

We will now be considering the completely state controllable system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} G \\ H \end{bmatrix} u \qquad u \in R^P. \tag{A1}$$

If the open loop eigenvalues satisfy (1.21), then we can apply transformation (1.14) which transforms (A1) into

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A-BP & B \\ 0 & D+PB \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} + \begin{bmatrix} G \\ H+PG \end{bmatrix} u \tag{A2}$$

where P is obtained using either (1.17) or (2.135). The transformation involved here can be written as

$$\begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{A3}$$

which possesses the explicit inverse

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}. \tag{A4}$$

Observe now that the pair (D+PB, H+PG) spans only the "fast" controllable subspace. Thus, design a feedback gain F such that

$$(D^* + H^*F) \qquad D^* = D + PB$$
$$H^* = H + PG$$

has M desired poles.

The control is of the form

$$u = F\eta \tag{A5}$$
$$= F(Py + z)$$
$$= [FP : F]\begin{bmatrix} y \\ z \end{bmatrix} \tag{A6}$$

and the resulting closed-loop system has N eigenvalues according to

$$\sigma(A-BP) \tag{A7}$$

and M eigenvalues according to

$$\sigma(D^* + H^*F). \tag{A8}$$

Now apply transformation (1.45) to (A1). This gives

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A-\hat{P}C & 0 \\ C & D+C\hat{P} \end{bmatrix}\begin{bmatrix} \xi \\ z \end{bmatrix} + \begin{bmatrix} G-\hat{P}H \\ H \end{bmatrix}u$$

where $\hat{P}$ is obtained using either (1.49) or (2.152). The transformation involved here can be written as

$$\begin{bmatrix} \xi \\ z \end{bmatrix} = \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \tag{A9}$$

which has the explicit inverse

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} I & \hat{P} \\ 0 & I \end{bmatrix}\begin{bmatrix} \xi \\ z \end{bmatrix} \tag{A10}$$

Observe now that the pair $(A-\hat{P}C,\ G-\hat{P}H)$ spans only the slow controllable subspace. Thus, design a feedback F such that

$$(A^* + G^*F) \qquad A^* = A-\hat{P}C$$
$$G^* = D+C\hat{P}$$

has N desired poles.

The control is of the form

$$u = F\xi \tag{A11}$$
$$= F(y-\hat{P}z)$$
$$= [F \vdots -F\hat{P}]\begin{bmatrix} y \\ z \end{bmatrix} \tag{A12}$$

and the resulting closed loop system has N eigenvalues of

$$\sigma(A^* + G^*F)$$

and M eigenvalues of

$$\sigma(D + C\hat{P}).$$

In general, both slow and fast modes may be designed for. In this case a general two-state design procedure may be implemented. Assume we have used either (1.83) or (1.94) on (A1) and thus have the form

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} G^* \\ H^* \end{bmatrix} u. \tag{A13}$$

We arbitrarily chose to design for the slow subsystem first. Thus, we chose an F_s such that

$$\sigma(A^* + G^*F_s)$$

has N desired "slow" eigenvalues. Letting

$$u = u_s + u_f$$

where now

$$u_s = F_s x,$$

the partially closed loop system looks like

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A^* + G^*F_s & 0 \\ H^*F_s & D^* \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} G^* \\ H^* \end{bmatrix} u . \tag{A14}$$

Now, let

$$v = w + Sx \tag{A15}$$

which transforms (A14) into

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A^* + G^*F_s & 0 \\ H^*F_s + s(A^*+G^*F_s)-D^*s & D^* \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} G^* \\ H^* + SG^* \end{bmatrix} u. \tag{A16}$$

We pick S such that

$$H^*F_s + S(A^* + GF_s) - D^*S = 0. \tag{A17}$$

This Lyapunov type equation has a unique solution if

$$\sigma(D^*) \cap \sigma(A^* + GF_s) = \phi \tag{A18}$$

and thus can be solved algebraically [12]. If

$$\inf|\sigma(D^*)| > \sup|\sigma(A^* + G^*F_s)|. \tag{A19}$$

Then the iterative scheme (1.92) may be used to solve (A17). With this S, the pair (D^*, H^*+SG^*) now spans only the fast controllable subspace, and we can design a feedback gain F_f such that

$$\sigma(D^* + (H^*+SG^*)F_f)$$

has M desired eigenvalues.

Thus, our composite control is

$$\begin{aligned} u &= F_s x + F_f v \\ &= G_s x + F_f(w + Sx) \\ &= (F_s + F_f s)x + F_f w \\ &= [(F_s + F_f s) : F_f]\begin{bmatrix} x \\ w \end{bmatrix} \end{aligned} \tag{A20}$$

and using either transformation (1.84) or (1.95), (A20) can be expressed in terms of our original state variables. This control place N eigenvalues of

$$\sigma(A^* + G^* F_s) \tag{A21}$$

and M eigenvalues of

$$\sigma(D^* + (H^* + SG^*)F_f). \tag{A22}$$

This technique has been applied to singularly perturbed systems [22] where it is shown to be a generalization to results obtained in [17,18,19]. This technique is also applicable to discrete time models as shown in [24]. In this case, the dominant eigenvalues are part of the "slow" spectrum.

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